From the pion to the dipion light-cone distribution amplitudes and the phenomena

Shan Cheng (HNU)

arXiv:2209.13312, Jian Chai, **SC** and Jun Hua
arXiv:2007.05550, **SC**, A. Khodjamirian and A V. Rusov
arXiv:1901.06071, **SC**

Seminar at University of Science and Technology of China
June 21, 2023, Hefei

Overview

1. Motivation
   - Light-cone distribution amplitudes
2. Pion LCDAs from the form factors
   - Modulus representation of dispersion relation
   - Data-driven extracting of a_n^π and m_0^π
3. Dipion LCDAs and the phenomena
   - Double expansion and the coefficient $B_{nl}^{(I)}(s)$
   - $B \to (\rho \to)\pi\pi$ and $D_s \to (f_0 \to) [\pi\pi]_\mathrm{S}$ form factors
4. Conclusion

Motivation

- Among the general coordinate transformations of the 4-dimensional Minkowski space that conserve the interval $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$, there are transformations only change the scale of the metric $g'_{\mu\nu}(x') = \omega(x)g_{\mu\nu}(x)$ and, consequently, preserve the angles and leave the light-cone invariant.

- conformal transformation correspond to a genaralization of the usual Poincaré group △ the conformal algebra in four-dimension has 15 generators: translations $\mathbf{P}_\mu$, Lorentz rotation $\mathbf{M}_{\mu\nu}$, dilatation $\mathbf{D}$ and special conformal translation $\mathbf{K}_\mu$, △ 10-parameter Lie algebra of the Poincaré group: $\mathbf{P}_\mu$, $\mathbf{M}_{\mu\nu}$

† examples △ dilatation (global scale transformation) $x^\mu \to x'^\mu = \lambda x^\mu$ and inversion $x^\mu \to x'^\mu = x^\mu/x^2$
△ special conformal transformation (sequential inversion) $x^\mu \to x'^\mu = (x^\mu + a^\mu x^2)/(1 + 2a \cdot x + a^2 x^2)$

- the generators act on a generic fundamental field $\Phi(x)$ with an arbitrary spin

$$\delta_P^\mu \equiv i[\mathbf{P}^\mu, \Phi(x)] = \partial^\mu \Phi(x), \quad i[\mathbf{M}^{\mu\nu}, \Phi(x)] = (x^\mu \partial^\nu - x^\nu \partial^\mu - \Sigma^{\mu\nu})\Phi(x),$$
$$i[\mathbf{D}, \Phi(x)] = (x \cdot \partial + l)\Phi(x), \quad i[\mathbf{K}^\mu, \Phi(x)] = (2x^\mu x \cdot \partial + x^2 \partial^\mu + 2l x^\mu - 2x_\nu \Sigma^{\mu\nu})\Phi(x).$$

△ $\Sigma^{\mu\nu}$ is the generator of spin rotations: $\Sigma^{\mu\nu}\phi = 0$, $\Sigma^{\mu\nu}\psi = 1/2\sigma^{\mu\nu}\psi$, $\Sigma^{\mu\nu}A^\alpha = g^{\nu\alpha}A^\mu - g^{\mu\alpha}A^\nu$
△ l is the scaling dimension which specifies the field transformation under the dilatations, △ $l = l^{\text{can}}$ in the free theory (classical level) $\Leftarrow$ the action of the theory is dimensionless, △ $l \neq l^{\text{can}}$ in the quantum theory, called the anomalous dimension)

Motivation

- an ultra-relativistic particle (quark or gluon) propagates close to the light-cone
- introduce the projections on the two independent light-like vectors $n_\mu, \bar{n}_\mu$

  △ $A_\mu, \quad A_+ = A_\mu n^\mu, \; A_- = A_\mu \bar{n}^\mu, \; A^2 = 2A_+A_- - A_\perp^2,$ △ $g_\perp^{\mu\mu} = g^{\mu\nu} - n^\mu \bar{n}^\nu - n^\nu \bar{n}^\mu$
- consider the special conformal transformation with $a_\mu = a\bar{n}_\mu$

  △ then $x_- \to x'_- = x_-/(1+2ax_-)$, this transformation map the light-ray in the x_- direction into itself

  △ together with the translations and dilatations along the same direction, $x_- \to x_- + c$, $x_- \to \lambda x_-$, form a collinear subgroup ($SL(2, R)$) of the full conformal group
- in the parton model, hadrons states are replaced by a bunch of collinear partons ($\bar{n}_\mu$), only to consider the quantum field "living" on the light-ray $\Phi(x) \to \Phi(\alpha n)$
- conformal group is reduced to the collinear subgroup that generates projective transformation on the line

$$\alpha \to \alpha' = \frac{a\alpha + b}{c\alpha + d}, \; ad - bc = 1; \quad \Phi(\alpha) \to \Phi'(\alpha) = (c\alpha + d)^{-2j}\Phi\left(\frac{a\alpha + b}{c\alpha + d}\right), \; j = \frac{l+s}{2}.$$

  △ generated by the four generators $\mathbf{P}_+$, $\mathbf{M}_{-+}$, $\mathbf{D}$, and $\mathbf{K}_-$, a collinear subalgebra of the conformal algebra

  △ introduce $\mathbf{L}_\pm = \mathbf{L}_1 \pm i\mathbf{L}_2 = -i\mathbf{P}_+(\mathbf{K}_-/2)$, $\mathbf{L}_0(\mathbf{E}) = i/2(\mathbf{D} \pm \mathbf{M}_{-+})$ satisfying $[\mathbf{L}_0, \mathbf{L}_\mp] = \mp\mathbf{L}_\mp$ and $[\mathbf{L}_-, \mathbf{L}_+] = -2\mathbf{L}_0$

  △ obtain $[\mathbf{L}_\pm, \Phi(\alpha)] = -\partial_\alpha(\alpha^2\partial_\alpha + 2j_\alpha)\Phi(\alpha) = L_\pm\Phi(\alpha)$ and $[\mathbf{L}_0, \Phi(\alpha)] = (\alpha\partial_\alpha + j)\Phi(\alpha) = L_0\Phi(\alpha)$ satisfying $[L_0, L_\mp] = \pm L_\mp$ and $[L_-, L_+] = 2L_0$ △ the algebra of $SL(2, R) \sim O(2, 1)$
- another subgroup corresponding to trans. of the 2-dimensional transverse plane

Motivation-LCDA

- the remaining generator **E** count the collinear twist of the field Φ

  △ $[\mathbf{E}, \Phi(\alpha)] = (l - s)/2\Phi(\alpha)$, △ commutes with all $\mathbf{L}_i$

- hadrons are described by LCDAs at different (collinear) twist, PDA, PDF

  △ collinear twist: dimension - spin projection on the plus-direction △ geometric twist: dimension - spin

- ie., $|\pi\rangle = \psi_{q\bar{q}}|q\bar{q}\rangle + \psi_{q\bar{q}g}|q\bar{q}g\rangle + \psi_{q\bar{q}q\bar{q}}|q\bar{q}q\bar{q}\rangle + \cdots$

$$\psi_\pi^n(x_i, k_{\perp i}, \lambda_i) = \langle n, x_i, k_{\perp i}, \lambda_i|\pi\rangle$$

† large Q^2, $k_\perp$ can been neglected/integrated $\psi_\pi^n(x_i, \lambda_i)$, separate out the spin to obtain the LCDA $\phi_\pi^n(x_i, Q)$

† corrections $\mathcal{O}(k_\perp^2/Q^2, m^2/Q^2, \alpha_s)$, scale dependence, RGE with the general solution in terms of Gegenbauer polynomials. ie.,

$$\text{leading twist} \quad \phi_\pi(x, \mu) = 6x(1-x)\sum_{n=0} a_n^\pi(\mu) C_n^{3/2}(x)$$

$$\text{twist three} \quad \phi_\pi^p(x, \mu) = \frac{m_0^\pi(\mu)}{P^+}\left[1 + 30\eta_{3\pi} C_2^{1/2}(x) - 3\eta_{3\pi}\omega_{3\pi} C_4^{1/2}\right]$$

$$\text{twist three} \quad \phi_\pi^\sigma(x) = \frac{m_0^\pi(\mu)}{P^+} 6x(1-x)\left[1 + 5\eta_{3\pi} C_2^{3/2}\right]$$

△ asymptotic behavior $a_0^\pi = f_\pi$, △ $a_{n\geqslant 2}^\pi(\mu)$ and $m_0^\pi(\mu)$ are obtained by non-pert. theory/lattice QCD

△ fine structure of pion (hadron), △ electromagnetic FFs, $B \to \pi\pi$; radiative and semileptonic decays $\to$ NP

Motivation-LCDA-a_2^π

- status of a_n^π study $\qquad \phi_\pi(x,\mu) = 6x(1-x)\sum_{n=0} a_n^\pi(\mu) C_n^{3/2}(x)$

† in QCD $\qquad a_n^\pi(\mu) = \langle \pi | q(z)\bar{q}(z) + z_\rho \partial_\rho q(z)\bar{q}(z) + \cdots |0\rangle$

† LQCD: $a_2^\pi(1\,\mathrm{GeV}) = 0.334 \pm 0.129$[RBC&UKQCD 2010] , 0.135 ± 0.032[RQCD 19]

△ a_4^π is not available ← the growing number of derivatives in $q\bar{q}$ operator

△ new technique is being developed[RQCD 2017, 2018].

† QCDSR: $a_2^\pi = 0.19 \pm 0.06$[Chernyak 1984], $0.26^{+0.21}_{-0.09}$[Khodjamirian 2004], 0.28 ± 0.08[Ball 06],

△ nonlocal vacuum condensate is introduced and modeled for $a_{n>2}^\pi$ [Bakulev 2001]

△ QCD sum rules as an inverse problem[Li 2020, Yu 2022]

quark-hadron duality → Legendre expansion of spectral density

† LCSRs: data-driven

△ $F_{B\to\pi}$: $a_2^\pi = 0.19 \pm 0.19$[Ball 2005], 0.16[Khodjamirian 2011], large error from B meson

△ $F_{\pi\gamma\gamma^*}$: $a_2^\pi = 0.14$[Agaev 2010] BABAR+CLEO, 0.10[Agaev 2012] Belle+CLEO,

large uncertainty of $a_{n>2}^\pi$, discrepancy at large Q^2, BESIII ?

△ F_π: $a_2^\pi = 0.24 \pm 0.17$[Bebek 1978] Wilson Lab+NA7, 0.20 ± 0.03[Agaev 2005] Wilson Lab+JLab,

large uncertainty of $a_{n>2}^\pi$, precise measurement in the small transfers

Motivation-LCDA-m_0^π

- status of m_0^π study $\phi_\pi^{p/\sigma}(x,\mu) \propto m_0^\pi(\mu)/P^+$

† in QCD $\langle \pi^+|\bar{u}(0)(-i\gamma_5)d(0)|0\rangle = f_\pi m_0^\pi(\mu), \quad m_0^\pi = m_\pi^2/(m_u+m_d)$

† $m_0^\pi(1\,\mathrm{GeV}) = 1.892$ GeV is obtained from $\chi\mathrm{PT}$[Leutwyler, 1996]

† $\phi_\pi^{p/\sigma}(m_0^\pi)$ are not involved in F_π^{LCSRs} due to the chiral symmetry limit

† give dominant contribution in F_π^{pQCD} due to the chiral enhancement $\mathcal{O}(m_0^\pi/Q^2)$

△ usually chosen at a fixed value in the previous pQCD study

Table 1. Input of m_0^π in the previous pQCD calculations.

Physical quantity (Accuracy)	m_0^π	Refs
Pion EM FF (NLO, 2p, twist-3)	1.74	[12–15]
Pion EM FF (NLO, 3p, twist-3)	1.74	[16]
Pion EM FF (NLO, 3p, twist-4)	1.90(1 GeV)	[9]
$B \to \pi$ FF (LO, 3p)	1.4	[17]
$B \to \pi$ FF (twist-2 NLO, 2p)	$1.74^{+0.67}_{-0.38}$	[18]
$B \to \pi$ FF (twist-3 NLO, 2p)	1.4	[19]

△ maybe the largest error source of pQCD predictions of F_π

△ the corresponding large uncertainty is formerly disregarded

Motivation-LCDA-F_π

- data-driven extraction of a_2^π, m_0^π from $F_\pi(q^2 < 0)$

† $F_\pi^{\rm LCSRs}(q^2)$ is applicable when $q^2 \in [-10, -1]$ GeV^2[Braun 1994, 2000, Bijnens 02]

$$F_\pi^{\rm LCSRs}(q^2) = F_\pi^{\rm t2,LO} + F_\pi^{\rm t2,NLO} + F_\pi^{\rm t4,LO} + F_\pi^{\rm t6,LO}$$

† $F_\pi^{\rm pQCD}(q^2)$ is applicable when $q^2 < -10$ GeV^2[Li 2001, Li 12, **SC** 14]

$$F_\pi^{\rm pQCD}(q^2) = F_\pi^{\rm t2,LO+NLO} + F_\pi^{\rm t3,LO+NLO} + F_\pi^{\rm t2\otimes t4,LO} + F_\pi^{\rm 3p,LO}$$

† data is only available in the resonant region $|q^2| \leqslant 2.5\,{\rm GeV}^2$

† the mismatch destroys the direct extracting programme from spacelike form factor

△ large errors $a_{n>2}$ terms become important in the intermediate and large transfers in LCSRs/*powerlessness*

† timelike form factor $F_\pi(q^2 > 0)$ provides another opportunity

△ BABAR: $e^+e^- \to \pi^+\pi^-(\gamma)$, $4m_\pi^2 \leqslant q^2 \lesssim 9$ GeV^2[BABAR 2012]

△ Belle: $\tau \to \pi\pi\nu_\tau$, $4m_\pi^2 \leqslant q^2 \leqslant 3.125$ GeV^2[Belle 2008]

△ BESIII: $e^+e^-(\gamma) \to \pi^+\pi^-$, $0.6 \leqslant Q^2 \leqslant 0.9\,{\rm GeV}^2$ with ISR[BESIII 2016]

- timelike measurement and spacelike predictions are related by dispersion relation

Pion LCDAs from the form factors

Dispersion relation

- dispersion relation is written in the integral over invariant mass $[4m_\pi^2, \infty)$
- the standard dispersion relation

$$F_\pi(q^2 < 0) = \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\mathrm{Im} F_\pi(s)}{s - q^2 - i\epsilon}$$

† the measurement is $|F_\pi(s)|^2$ rather than $\mathrm{Im} F_\pi(s)$

† model dependence in $F_\pi^{\mathrm{data}}(s)$ parameterization to reproduce $|F_\pi^{\mathrm{data}}(s)|$

† vector dominate model (VDM) to parameterize the data[BABAR 2012]

$$F_\pi^{data}(s) = \sum_{n=0,\cdots}^{N} \frac{c_n^\pi \, BW_{\rho_n}^{GS}(s)}{c_n^\pi}, \qquad c_0^\pi \to \frac{1 + c_\omega^\pi \, BW_\omega^{KS}(s)}{1 + c_\omega^\pi},$$

$$BW_{\rho_n}^{GS}(s) = \frac{m_n^2 + m_n \Gamma_n d(m_n)}{m_n^2 - s + f(s) - i\sqrt{s}\Gamma_n(s)}, \qquad BW_\omega^{KS}(s) = \frac{m_\omega^2}{m_\omega^2 - s - i\, m_\omega \Gamma_\omega}.$$

△ Gounaris-Sakuria and Kühn-Santamaria representations[Gounaris 1968, Kühn 1990]

△ $N = 4$ & $\rho - \omega$ interaction, the data is described by 18 parameters

† The application of $F_\pi^{data}(s)$ at high energy tails is not physical

△ resonances above $N = 4$ are not included, △ $F_\pi^{data}(s \to \infty) \nrightarrow 1/s$

Dispersion relation—modulus representation

- the modulus representation of dispersion relation[SC, A. Khodjamirian and A. Rosov 2020]

$$\frac{\ln F_\pi(q^2)}{q^2\sqrt{s_0-q^2}} = \frac{1}{2\pi}\int_{s_0}^{\infty}\frac{ds\,\ln|F_\pi(s)|^2}{s\sqrt{s-s_0}\,(s-q^2)}, \qquad q^2<s_0$$

$$\Updownarrow \qquad F_\pi(q^2) = \exp\left[\frac{q^2\sqrt{s_0-q^2}}{2\pi}\int_{s_0}^{\infty}\frac{ds\,\ln|F_\pi(s)|^2}{s\sqrt{s-s_0}\,(s-q^2)}\right], \qquad q^2<s_0$$

† introduce an auxiliary function[Geshkenbein 1998]

$$g_\pi(q^2) \equiv \frac{\ln F_\pi(q^2)}{q^2\sqrt{s_0-q^2}}$$

† implement Cauchy theorem and Schwartz reflection principle on g_π

† the only assumption in the derivation: $F_\pi(q^2)$ is free from zeros in the complex q^2 plane, then $\ln F_\pi(q^2)$ does not diverge[Leutwyler 2002, Ananthanarayan 2011]

$\triangle$ if $F_\pi(q^2)$ has zeros in the complex q^2 plane, deserves a separate analysis [Dominguez 2001, Ananthanarayan 2004]

$\triangle$ $F_\pi(q^2)$ evaluated by the standard and modulus DR have a tiny difference

$\rightarrow$ the zeros of $F_\pi(q^2)$ are either absent or their influence is beyond our accuracy

Timelike FF from data and dual-resonance model

$$F_\pi(q^2) = \exp\left[\frac{q^2\sqrt{s_0 - q^2}}{2\pi}\int_{s_0}^{\infty}\frac{ds\,\ln|F_\pi(s)|^2}{s\sqrt{s-s_0}\,(s-q^2)}\right], \quad q^2 < s_0$$

$$|F_\pi(s)| = \Theta(s_{\rm max} - s)\,|F_\pi^{\rm (data)}(s)| + \Theta(s - s_{\rm max})\,|F_\pi^{\rm (tail)}(s)|$$

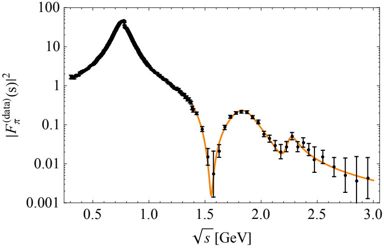
- the dual-resonance models with $N_c = \infty$ limit of QCD[Dominguez 2002, Bruch 2004]

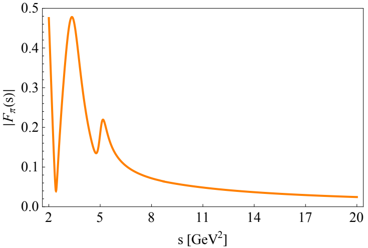
$$F_\pi^{(tail)}(s) = F_\pi^{(dQCD)}(s) = \sum_{n=0}^{\infty} c_n\, BW_n(s), \quad c_n = \frac{(-1)^n\Gamma(\beta - 1/2)}{\alpha' m_n^2\sqrt{\pi}\,\Gamma(n+1)\Gamma(\beta-1-n)}, \quad BW_n(s) = BW_n^{KS}(s)$$

△ $m_n^2 = m_\rho^2(1+2n)$, $\alpha' = 1/2m_\rho^2$, $\Gamma_n = \gamma\, m_n$, $\gamma = 0.193$ is adjusted to the total width of $\rho(770)$

△ Matching $|F_\pi^{(data)}(s_{max})| = |F_\pi^{(tail)}(s_{max})|$ indicates $\beta = 2.09 \pm 0.13$

△ reproduce $F_\pi^{\rm (dQCD)}(0) = 1$ and $\lim_{s\to-\infty} F_\pi^{\rm (dQCD)}(s) \sim 1/s^{\beta-1}$





- LCSRs prediction of $F_\pi(q^2)$

$\phi_\pi(x,\mu) = 6x(1-x)\sum_{n=0} a_n^\pi(\mu) C_n^{3/2}(x)$

△ energetic pion [Braun 1999, Bijnens 2002]

$$F_\pi^{(\mathrm{LCSR})}(Q^2) = F_\pi^{(\mathrm{as})}(Q^2) + \sum_{n=2,4,..} a_n(\mu_0)\, f_n(Q^2,\mu,\mu_0)$$

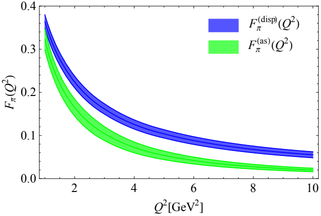
$$F_\pi^{(\mathrm{as})}(Q^2) = F_\pi^{(\mathrm{tw2,as})}(Q^2) + F_\pi^{(\mathrm{tw4,\,LO})}(Q^2) + F_\pi^{(\mathrm{tw6,\,fact})}(Q^2)$$

△ f_n is the integral of Gegenbauer polynomials with the Borel exponent

△ soft dynamics dominated, one quark carries almost the whole momentum

△ $F_\pi^{\mathrm{LCSRs}}(q^2)$ in terms of LCDAs, Gegenbauer moments dependence

△ leading asymptotic term reproduces the asymptotic pQCD behavior



△ $F_\pi^{(\mathrm{disp})}(q^2) = \exp\left[\frac{q^2\sqrt{s_0-q^2}}{2\pi}\int_{s_0}^{\infty}\frac{ds\ \ln|F_\pi(s)|^2}{s\sqrt{s-s_0}\,(s-q^2)}\right]$

△ significant gap between $F_\pi^{(\mathrm{disp})}$ and $F_\pi^{(\mathrm{as})}$

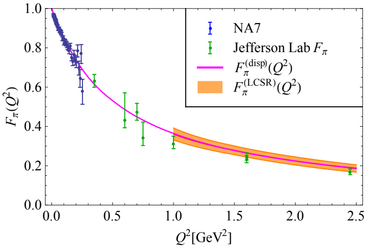
△ the 2nd term $\propto a_n$ gives significant effect

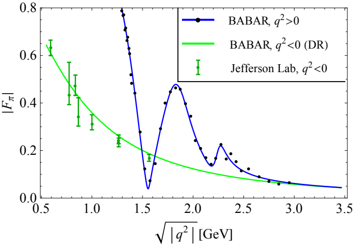
Result of a_n^π and $F_\pi(q^2)$

- χ^2 fit to reveal the more inner structures of pion meson

$$\chi^2 = \sum_{i=1}^{N_p} \frac{1}{\sigma_i^2} \left[\sum_{n=2,4,..}^{n_{\max}} a_n(\mu_0)\, f_n(Q_i^2, \mu_0) + F_\pi^{(\rm as)}(Q_i^2) - F_\pi^{(\rm disp)}(Q_i^2) \right]^2$$

Model	$a_2(1\,\mathrm{GeV})$	$a_4(1\,\mathrm{GeV})$	$a_6(1\,\mathrm{GeV})$	$\chi^2_{\min}/\mathrm{ndf}$
$\{a_2\}$	0.302 ± 0.046			4.08
$\{a_2, a_4\}$	0.279 ± 0.047	0.189 ± 0.060		0.75
$\{a_2, a_4, a_6\}$	0.270 ± 0.047	0.179 ± 0.060	0.123 ± 0.086	0.073





△ $|F_\pi(s)| \sim |F_\pi^{(\rm disp)}(|q^2|)|$ at $|\sqrt{q^2}| \gtrsim 3\,\mathrm{GeV}$, manifests analyticity of the modulus representation

$$F_\pi(q^2) = \exp\left[\frac{q^2\sqrt{s_0-q^2}}{2\pi}\int_{s_0}^{\infty}\frac{ds\,\ln|F_\pi(s)|^2}{s\sqrt{s-s_0}\,(s-q^2)}\right],\quad q^2<s_0$$

$$|F_\pi(s)| = \Theta(s_{\rm max}-s)\,|F_{\pi,{\rm Inter}}^{\rm (data)}(s)| + \Theta(s-s_{\rm max})\,|F_\pi^{\rm (pQCD)}(s)|$$

- interpolating the data with evenly distribution under the interval 0.01 GeV

  △ data sample densities are roughly 0.01 GeV, 0.002 GeV and 0.1 GeV in the near resonances, resonances located and away resonances regions, respectively

- pQCD prediction of high energy tail[SC 2019]

  △ An unique advantage of pQCD is that the high energy tail can be directly calculated by perturbative theorem

† pQCD calculation based on k_T factorization theory (take spacelike FF for example)

$$\langle\pi^-(p_2)|J_\mu^{\rm em}|\pi^-(p_1)\rangle = \oint dz_1dz_2\,H^{ijkl}_{\gamma\beta\alpha\delta}(z_2,z_1)\langle\pi^-(p_2)\Big|\{\bar{d}_\gamma(z_2)e^{\left[ig_s\int_0^{z_2}d\sigma_{\nu'}A_{\nu'}(\sigma)\right]}u_\beta(0)\}_{kj}\Big|0\rangle_{\mu_t}$$

$$\langle 0\Big|\left\{\bar{u}_\alpha(0)\,e^{\left[ig_s\int_{z_1}^0 d\sigma_\nu A_\nu(\sigma)\right]}d_\delta(z_1)\right\}_{il}\Big|\pi^-(p_1)\rangle_{\mu_t}$$

△ Seperation: SD (propagator) and LD (Heisenberg operator) △ Hard kernel and nonlocal MEs ⇐ spin structures (Fierz trans.) △ nonlocal MEs, defined in terms of LCDAs by twist △ Truncated (factorizable) scale μ_t

FFs from pQCD

- hard kernel associated with the lowest Fock state

$$H^{ijkl}_{\gamma\beta\alpha\delta}(z_1, z_2) = (-1)\,[ig_s\gamma_m]_{\alpha\beta}\; T^{ij}\;\left[(ie_q\gamma_\mu)S_0(0-z_1)(ig_s\gamma_n)\right]_{\gamma\delta}\; T^{kl}\;\left[-iD^0_{mn}(z_1-z_2)\right]$$

- the free propagators in the coordinate space

$$S_0(z) = \frac{i}{2\pi}\,\frac{\not z}{z^4}\,, \qquad D^0_{mn}(z) = \frac{1}{4\pi}\,\frac{g_{mn}}{z^2}$$

- about the nonlocal MEs, different spin structures (LCDAs at different twists)

$$\langle 0\left|\;\{\overline{u}_\alpha(z_1)\,[z_1,0]\,d_\delta(0)\}_{il}\;\right|\pi^-(p_1)\rangle_{\mu_t} = \frac{\delta_{il}}{3}\left\{\frac{1}{4}\,(\gamma_5\gamma^\rho)_{\delta\alpha}\,\langle 0|\overline{u}(z_1)\,[z_1,0]\;(\gamma_\rho\gamma_5)\;d(0)|\pi^-(p_1)\rangle_{\mu_t}\right.$$
$$\left. +\;\frac{1}{8}\left(\sigma^{\tau\tau'}\gamma_5\right)_{\delta\alpha}\,\langle 0|\overline{u}(z_1)\,[z_1,0]\;(i\sigma_{\tau\tau'}\gamma_5)\;d(0)|\pi^-(p_1)\rangle_{\mu_t} + \cdots\right\}$$

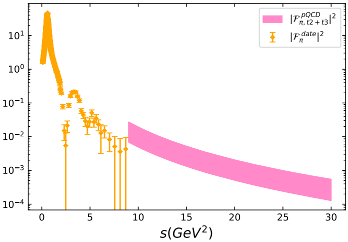
- high Fock states contribution, like three particle configuration $q\bar{q}g$
- pQCD prediction can be arranged as

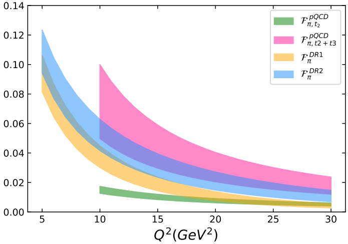
$$\mathcal{F}^{\rm pQCD}_\pi(q^2) = (m^\pi_0)^2 F_1(q^2) + m^\pi_0 F_2(q^2) + F_3(q^2)$$
$$+ m^\pi_0 a^\pi_2 F_4(q^2) + a^\pi_2 F_5(q^2) + (a^\pi_2)^2 F_6(q^2)$$

△ the function F_3 collects the contributions from the asymptotic term and partial high twists terms

△ power hiearachy between different terms, like $(m^\pi_0)^2 F_1 > m^\pi_0 F_2(q^2) > F_3(q^2)\cdots$

Result of m_0^π, a_n^π and $F_\pi(q^2)$





△ timelike (left) and spacelike (right) form factors

△ the pQCD predictions are obtained by taking $m_0^\pi(1\,\mathrm{GeV}) = 1.6 \pm 0.4\,\mathrm{GeV}$ and $a_2^\pi(1\,\mathrm{GeV}) = 0.25 \pm 0.25$

△ the effect from the scale evolutions are also taken in to account

† $F^{pQCD}_{\pi,t2+t3}(s)$ marries with the data at the intermediate regions within uncertainty

† the chiral enhancement effect is significant △ the large gap between $F^{pQCD}_{\pi,t2}$ and F^{DR}_π

† high energy tail gives large contribution to $F^{\mathrm{DR}}_\pi(Q^2)$, especially in the large q^2

△ the logarithm expression of the timelike FF strengthens the role of high energy tail in the dispersion relation

† the fit of F^{pQCD}_π with F^{DR}_π can not arrive at a good result of a_2^π, but m_0^π

△ the uncertainty of F^{DR1}_π is larger than the leading twist pQCD prediction

Result of m_0^π, a_n^π and $F_\pi(q^2)$

- χ^2 fit to reveal the more inner structures of pion meson

$$\chi^2 = \sum_{i=1}^{11} \frac{\left[\mathcal{F}_\pi^{\rm DR2}(Q_i^2) - \mathcal{F}_\pi^{\rm pQCD}(Q_i^2)\right]^2}{\left[\delta\mathcal{F}_\pi^{\rm DR2}(Q_i^2)\right]^2}$$

△ iteration with the initial inputs $m_0^\pi(1\,{\rm GeV}) = 1.6 \pm 0.4\,{\rm GeV}$, $a_2^\pi(1\,{\rm GeV}) = 0.25 \pm 0.25$

Scenario	I	II
m_0^π (GeV)	$1.37^{+0.29}_{-0.32}$	$1.31^{+0.27}_{-0.30}$
a_2^π	0.25 ± 0.25	0.23 ± 0.25

△ fitting results of m_0^π and a_2^π at the default scale 1 GeV

△ Scenario I (II) represents the fit with(out) considering the scale running of nonperturbative parameters

- about the first gegenbauer coefficient a_2^π

† LQCD: 0.334 ± 0.129[UKQCD 2010], 0.155 ± 0.035[RQCD 2019], $0.258^{+0.079}_{-0.052}$[LPC 2022]

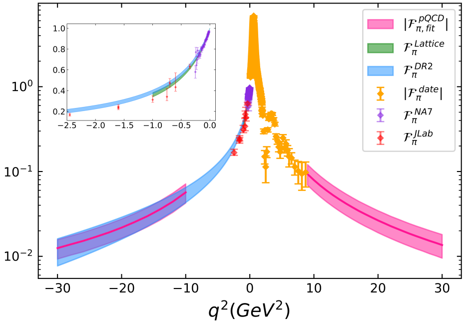
† data-driven determination from pion from factor: 0.279 ± 0.047 [SC 2020]

† $F_{\pi\gamma\gamma^*}$ is theoretical clear, measurements are discrepant from BABAR and Belle

† settle down with the foresee Belle-II and BESIII measurements ?

Result of m_0^π, a_n^π and $F_\pi(q^2)$

- † with the new result as inputs[Jian Chai, SC and Jun Hua 2022]



- † power hiearachy $(m_0^\pi)^2 F_1 > m_0^\pi F_2(q^2) > F_3(q^2) \cdots$
- † pQCD consists with BABAR data in the intermediate region much better
- † a good agreement in the large recoiled region $[-1, 0]$ GeV^2 within uncertainties

DiPion LCDAs and the phenomena

Why dipion LCDAs ?

- CKM matrix is a crucial criterion of the Standard Model[PDG 2022]

- long standing $|V_{ub}|$ tension

† $|V_{ub}| = (3.82 \pm 0.20) \times 10^{-3}$, mainly extracted from $B \to X_u l\nu$ and $B \to \pi l\nu$ decay

† $|V_{ub}|_{\rm incl} = (4.13 \pm 0.25) \times 10^{-3}$, $|V_{ub}|_{\rm excl} = (3.70 \pm 0.16) \times 10^{-3}$, $\sim 1.7 - 2.7\sigma$

† enlarge the set of exclusive processes to determine $|V_{ub}|$, a candidate is $B \to \rho l\nu$

△ ρ is reconstructed by $\pi\pi$ invariant mass spectral, width effect/nonresonant contribution ?

△ the underlying consideration is $B \to \pi\pi l\bar{\nu}_l$ (B_{l4}) [S. Faller 2014]

- Besides the unitarity triangles (orthogonality), the unitarity can also be tested by the normalization conditions, the least precisely determinations is

$$|V_{us}|^2 + |V_{cs}|^2 + |V_{ts}|^2 = 1.004 \pm 0.012, \quad |V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1.001 \pm 0.012$$

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9985 \pm 0.0007$$

† $|V_{cs}| = 0.975 \pm 0.006$, mainly extracted from the (semi)leptonic $D_{(s)}$ decays

† $|V_{cs}| = 0.972 \pm 0.007$, $|V_{cs}| = 0.984 \pm 0.012$, $\sim 1.5\sigma$ tension

† new channels like semileptonic D_s decays and are highly anticipated

△ problems encountered, $D_s \to f_0 l\nu$ has large uncertainty due to the width and complicate structure

- we need to study dipion and dikaon LCDAs

Dipion DAs

- Chiral-even and odd LC expansion with gauge factor $[x,0]$[Polyakov 1999, Diehl 1998]

$$\langle \pi^a(k_1)\pi^b(k_2)|\bar{q}_f(xn)\gamma_\mu \tau q_{f'}(0)|0\rangle = \kappa_{ab}\, k_\mu \int dx\, e^{izx(k\cdot n)}\, \Phi_{\parallel}^{ab,ff'}(z,\zeta,k^2),$$

$$\langle \pi^a(k_1)\pi^b(k_2)|\bar{q}_f(xn)\sigma_{\mu\nu}\tau q_{f'}(0)|0\rangle = \kappa_{ab}\, \frac{2i}{f_{2\pi}^{\perp}}\, \frac{k_{1\mu}k_{2\nu} - k_{2\mu}k_{1\nu}}{2\zeta - 1} \int dx\, e^{izx(k\cdot n)}\, \Phi_{\perp}^{ab,ff'}(z,\zeta,k^2).$$

△ $n^2 = 0$, △ index f, f' respects the (anti-)quark flavor, △ a, b indicates the electric charge of each pion,
△ coefficient $\kappa_{+-/00} = 1$ and $\kappa_{+0} = \sqrt{2}$, △ $k = k_1 + k_2$ is the invariant mass of dipion state,
△ $\tau = 1/2$, $\tau^3/2$ corresponds to the isoscalar and isovector 2πDAs,
△ higher twist proportional to $1, \gamma_\mu\gamma_5$ have not been discussed yet, γ_5 vanishes because of P-parity conservation

† three independent kinematic variables

△ momentum fraction z carried by anti-quark with respecting to the total momentum of dipion state, △ longitudinal momentum fraction carried by one of the pions $\zeta = k_1^+/k^+$, $2q\cdot\bar{k}\,(\propto 2\zeta - 1)$ △ the invariant mass squared k^2

† the chirally odd constant $f_{2\pi}^{\perp}$ is defined by the local matrix element

$$\lim_{k^2\to 0} \langle \pi^a(k_1)\pi^b(k_2)|\bar{q}(0)\sigma_{\mu\nu}\tau^3/2q(0)|0\rangle = 2i\,\epsilon^{abc}/f_{2\pi}^{\perp}(k_{1\mu}k_{2\nu} - k_{2\mu}k_{1\nu})$$

† normalization conditions

$$\int_0^1 \Phi_{\parallel(\perp)}^{I=1(0)}(u,\zeta,k^2) = (2\zeta - 1)F_\pi^{(t)}(k^2), \quad \int_0^1 dz\,(2z-1)\Phi_{\parallel}^{I=0}(z,\zeta,k^2) = -2M_2^{(\pi)}\zeta(1-\zeta)F_\pi^{\mathrm{EMT}}(k^2).$$

△ $F_\pi^{em}(0) = 1$, △ $F_\pi^{t}(0) = 1/f_{2\pi}^{\perp}$, △ $F_\pi^{\mathrm{EMT}}(0) = 1$,
△ $M_2^{(\pi)}$ is the momentum fraction carried by quarks in the pion associated to the usual quark distribution

Dipion DAs

- 2πDAs is decomposed in terms of $C_n^{3/2}(2z-1)$ and $C_\ell^{1/2}(2\zeta-1)$

$$\Phi^{I=1}(z,\zeta,k^2,\mu) = 6z(1-z)\sum_{n=0,\text{even}}^{\infty}\sum_{l=1,\text{odd}}^{n+1} B_{n\ell}^{I=1}(k^2,\mu)C_n^{3/2}(2z-1)C_\ell^{1/2}(2\zeta-1)$$

$$\Phi^{I=0}(z,\zeta,k^2,\mu) = 6z(1-z)\sum_{n=1,\text{odd}}^{\infty}\sum_{l=0,\text{even}}^{n+1} B_{n\ell}^{I=0}(k^2,\mu)C_n^{3/2}(2z-1)C_\ell^{1/2}(2\zeta-1)$$

- $B_{n\ell}(k^2,\mu)$ have similar scale dependence as the a_n of π,ρ,f_0 mesons

$$B_{n\ell}(k^2,\mu) = B_{n\ell}(k^2,\mu_0)\left[\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]^{\frac{\gamma_n^{(0)}-\gamma_0^{(0)}}{2\beta_0}}$$

$$\gamma_n^{\perp(\parallel),(0)} = 8C_F\left(\sum_{k=1}^{n+1}\frac{1}{k}-\frac{3}{4}-\frac{1}{2(n+1)(n+2)}\right)$$

- Watson theorem of $\pi-\pi$ scattering amplitudes △ implies an intuitive way to express the imaginary part of 2πDAs, △ leads to the Omnés solution of $N-$subtracted dispersion relation for the coefficients

$$B_{n\ell}^{I}(k^2) = B_{n\ell}^{I}(0)\,\mathrm{Exp}\left[\sum_{m=1}^{N-1}\frac{k^{2m}}{m!}\frac{d^m}{dk^{2m}}\ln B_{n\ell}^{I}(0)+\frac{k^{2N}}{\pi}\int_{4m_\pi^2}^{\infty}ds\,\frac{\delta_\ell^I(s)}{s^N(s-k^2-i0)}\right]$$

△ excellent description of pion form factor up to $k^2\sim 2.5$ GeV^2

△ 2πDAs in a wide range energies is given by δ_ℓ^I and a few subtraction constants

Dipion DAs

† soft pion theorem relates the chirarlly even coefficients with a_n^π

$$\sum_{\ell=1}^{n+1} B_{n\ell}^{\parallel, I=1}(0) = a_n^\pi \,, \quad \sum_{\ell=0}^{n+1} B_{n\ell}^{\parallel, I=0}(0) = 0$$

† 2πDAs relate to the skewed parton distributions (SPDs) in the pion by crossing

△ express the moments of SPDs in terms of $B_{nl}(k^2)$ in the forward limit as

$$M^\pi_{N=\mathrm{odd}} = \frac{3}{2}\frac{N+1}{2N+1} B^{I=1}_{N-1,N}(0), \quad M^\pi_{N=\mathrm{even}} = 3\frac{N+1}{2N+1} B^{I=0}_{N-1,N}(0)$$

† in the vicinity of the resonance, 2πDAs reduce to the DAs of ρ/f_0

△ relation between the a_n^ρ and the coefficients $B_{n\ell}$

$$a_n^\rho = B_{n1}(0)\,\mathrm{Exp}\left[\sum_{m=1}^{N-1} c_m^{n1} m_\rho^{2m}\right], \quad c_m^{(n1)} = \frac{1}{m!}\frac{d^m}{dk^{2m}}\left[\ln B_{n1}(0) - \ln B_{01}(0)\right]$$

△ f_ρ relates to the imaginary part of $B_{nl}(m_\rho^2)$ by $\quad \langle \pi(k_1)\pi(k_2)|\rho\rangle = g_{\rho\pi\pi}(k_1 - k_2)^\alpha \epsilon_\alpha$

$$f_\rho^\parallel = \frac{\sqrt{2}\,\Gamma_\rho\,\mathrm{Im}B_{01}^\parallel(m_\rho^2)}{g_{\rho\pi\pi}}, \quad f_\rho^\perp = \frac{\sqrt{2}\,\Gamma_\rho\, m_\rho\,\mathrm{Im}B_{01}^\perp(m_\rho^2)}{g_{\rho\pi\pi}\, f_{2\pi}^\perp}$$

† The subtraction constants of $B_{n\ell}(s)$ [SC 2019, 2023]

△ firstly studied in the effective low-energy theory based on instanton vacuum △ updated in 2019 and 2023

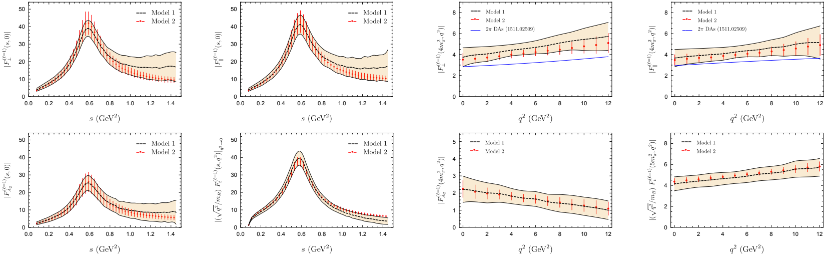
(nl)	$B^{\|\|}_{n\ell}(0)$	$c_1^{\|\|,(nl)}$	$\frac{d}{dk^2}\ln B^{\|\|}_{n\ell}(0)$	$B^{\perp}_{n\ell}(0)$	$c_1^{\perp,(nl)}$	$\frac{d}{dk^2}\ln B^{\perp}_{n\ell}(0)$
(01)	1	0	1.46 $\to$ 1.80	1	0	0.68 $\to$ 0.60
(21)	-0.113 $\to$ 0.218	-0.340	0.481	0.113 $\to$ 0.185	-0.538	-0.153
(23)	0.147 $\to$ -0.038	0	0.368	0.113 $\to$ 0.185	0	0.153
(10)	-0.556 $\to$ -0.300	-	0.413 $\to$ 0.375	-	-	-
(12)	0.556 $\to$ 0.300	-	0.413 $\to$ 0.375	-	-	-

Phenomena

- $B \to \pi\pi$ transition matrix element is defined in terms of the form factors

$$
\begin{aligned}
i\langle \pi^+(k_1)\pi^0(k_2)|\bar{u}\gamma_\nu(1-\gamma_5)b|\bar{B}^0(p)\rangle = {} & F_\perp(q^2,k^2,\zeta)\,\frac{2}{\sqrt{k^2}\sqrt{\lambda_B}}\, i\epsilon_{\nu\alpha\beta\gamma}\, q^\alpha\, k^\beta\, \bar{k}^\gamma \\
& + F_t(q^2,k^2,\zeta)\,\frac{q_\nu}{\sqrt{q^2}} + F_0(q^2,k^2,\zeta)\,\frac{2\sqrt{q^2}}{\sqrt{\lambda_B}}\left(k_\nu - \frac{k\cdot q}{q^2}q_\nu\right) \\
& + F_\parallel(q^2,k^2,\zeta)\,\frac{1}{\sqrt{k^2}}\left(\bar{k}_\nu - \frac{4(q\cdot k)(q\cdot\bar{k})}{\lambda_B}k_\nu + \frac{4k^2(q\cdot\bar{k})}{\lambda_B}q_\nu\right)
\end{aligned}
$$

- P-wave contribution to $B \to \pi\pi$ form factors with $I = 1$ dipion [SC, Khodjamirian, Vitor 2017]
- $P(S)$-wave contributions to $B \to \pi\pi$ form factors with $I = 1(0)$ dipion [SC 2019]



- $B_s \to (f_0 \to)KK$ form factors[SC and Jian-min Shen 2020]

Phenomena

- Semileptonic $D_{(s)}$ decays provide a clean environment to study scalar mesons
  - △ $D_{(s)} \to a_0 e^+ \nu$[BESIII 18, 21], $D^+ \to f_0/\sigma(\to \pi^+\pi^-)e^+\nu$[BESIII 19], $D_s \to f_0(\to \pi^+\pi^-)e^+\nu$[CLEO 09]
  - △ $D_s \to f_0(\to \pi^0\pi^0, K_s K_s)e^+\nu$ Branching ratio[BESIII 22], $D_s \to f_0(\to \pi^+\pi^-)e^+\nu$ form factor[BESIII 23]

$$\mathcal{B}(D_s \to f_0(\to \pi^0\pi^0)e^+\nu) = (7.9 \pm 1.4 \pm 0.3) \times 10^{-4}$$
$$\mathcal{B}(D_s \to f_0(\to \pi^+\pi^-)e^+\nu) = (17.2 \pm 1.3 \pm 1.0) \times 10^{-4}$$

  - △ isospin symmetry expectation $\mathcal{B}(f_0 \to \pi^+\pi^-)/\mathcal{B}(f_0 \to \pi^0\pi^0) = 2$, possible $\rho^0 \to \pi^+\pi^-$ pollution
  - △ $f_+^{f_0}(0)|V_{cs}| = 0.504 \pm 0.017 \pm 0.035$

- theoretical consideration $\frac{d\Gamma(D_s^+ \to f_0 l^+ \nu)}{dq^2} = \frac{G_F^2 |V_{cs}|^2 \lambda^{3/2}(m_{D_s}^2, m_{f_0}^2, q^2)}{192\pi^3 m_{D_s}^3} |f_+(q^2)|^2$
- f_0 is observed in the $\pi\pi$ invariant mass spectral
- improve the calculation by considering the width effect
  - † resonances model ($f_0 \to [\pi\pi]_{\rm S}$) [BESIII 23]

$$\frac{d\Gamma(D_s^+ \to [\pi\pi]_{\rm S}\, l^+\nu)}{ds dq^2} = \frac{1}{\pi} \frac{G_F^2 |V_{cs}|^2}{192\pi^3 m_{D_s}^3} |f_+(q^2)|^2 \frac{\lambda^{3/2}(m_{D_s}^2, s, q^2)\, g_1 \beta_\pi(s)}{|m_{\rm S}^2 - s + i\,(g_1\beta_\pi(s)) + g_2\beta_K(s))\,|^2}$$

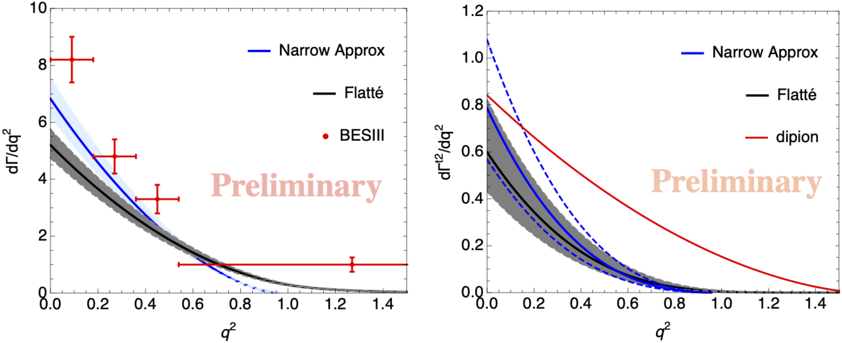
  - † stable $\pi\pi$ state

$$\frac{d^2\Gamma(D_s^+ \to [\pi\pi]_{\rm S}\, l^+\nu)}{dk^2 dq^2} = \frac{G_F^2 |V_{cs}|^2}{192\pi^3 m_{D_s}^3} \frac{\beta_{\pi\pi}(k^2)\sqrt{\lambda_{D_s}}\, q^2}{16\pi} \sum_{\ell=0}^{\infty} 2|F_0^{(\ell)}(q^2, k^2)|^2$$

- $D_s \to f_0$ ffs to $D_s \to [\pi\pi]_{\rm S}$ ffs [**SC** 2017,19,20, S. Descotes-Genon 19] in $B_{(s)}$ cases

Phenomena

- $D_s \to (f_0 \to) [\pi\pi]_S e\nu_e$ and $D_s^+ \to [\pi\pi]_S e^+\nu$ at leading twist[SC 2023, to be appear]



- significant model dependence, require a model independent study
- subleading twist contribution is indeed important △ go further to study the twist three 2πDAs
- significant difference in comparison to the result obtained from the resonant model
- further measurements would help us to understand the dipion system much better

Conclusion

- modulus representation of DR, LCSRs calculation + BABAR data,
  $a_2(1\,\mathrm{GeV}) = (0.22 - 0.33)\,, \quad a_4(1\,\mathrm{GeV}) = (0.12 - 0.25)$
  △ Pion deviates from the purely asymptotic one △ a_2^π is not enough, more inner structures
- modulus representation of DR, pQCD calculation + BABAR data,
  $m_0^\pi(1\,\mathrm{GeV}) = 1.37^{+0.29}_{-0.32}\,, \quad a_2(1\,\mathrm{GeV}) = 0.25 \pm 0.25$
- with leading twist dipion LCDAs, $B \to \pi\pi$ and $D_s \to [\pi\pi]_{\mathrm{S}}$ form factors are calculated, and compared to the result obtained from B meson LCSRs
- twist three dipion LCDAs are highly anticipated to improve prediction power
- Jefferson Lab 12 GeV upgrade program (9 GeV^2)
  to extract the nonperturbative paras (besides the lattice evaluation)
- BEPCII, up to 5.4 GeV in 2023-2024
- D_s/D_s^* events: BESIII $\mathcal{O}(10^6)$, Belle II $\mathcal{O}(10^9)$, STCF $\mathcal{O}(10^9)$, CEPC $\mathcal{O}(10^{10})$

Thank you for your patience.