Quark superfluidity in the two-fluid formalism


for a short summary, see arXiv:1212.4410 [hep-ph]

• Motivation: hydrodynamics of CFL
• Superfluids as two-component fluids
• Link microscopic physics with hydro
  – $T = 0$: one fluid
  – $T > 0$: two fluids
• **Motivation: hydrodynamics in compact stars**

• What are compact stars made of?
  
  Are they ...
  
  ... neutron stars?
  
  ... hybrid stars?
  
  ... quark stars?

• For various properties, need hydrodynamics:

  
  – *pulsar glitches* e.g., B. Link, MNRAS 422, 1640 (2012)
  
  – *magnetohydrodynamics* e.g., P. D. Lasky, B. Zink, K. D. Kokkotas, arXiv:1203.3590
  
Hefei, May 3, 2013

• Dense quark matter in compact stars – CFL (p. 1/3)

3-flavor, asymptotically dense matter

\[ (0 \approx m_s \approx m_u \approx m_d \ll \mu) \]

"color-flavor locked phase (CFL)"

M. Alford, K. Rajagopal, F. Wilczek, NPB 537, 443 (1999)

Cooper pair condensate

\[ \langle \psi_i^\alpha \psi_j^\beta \rangle \propto \epsilon^{\alpha\beta A} \epsilon_{ij}^A \]

\[ \Rightarrow \quad SU(3)_c \times SU(3)_L \times SU(3)_R \times U(1)_B \rightarrow SU(3)_{c+L+R} \times U(1)_Q \times U(1)_{\tilde{Q}} \times \mathbb{Z}_2 \]
• **Dense quark matter in compact stars – CFL (p. 2/3)**

CFL breaks chiral symmetry

\[
\langle \psi_R \psi_R \rangle, \langle \psi_L \psi_L \rangle, \text{however}
\]

\[
SU(3)_c \times SU(3)_L \times SU(3)_R \times U(1)_B \rightarrow SU(3)_{c+L+R} \times \mathbb{Z}_2
\]

• octet of pseudo-Goldstone modes \( K^0, K^\pm, \pi^0, \ldots \)

CFL is a (baryon) superfluid

\[
SU(3)_c \times SU(3)_L \times SU(3)_R \times U(1)_B \rightarrow SU(3)_{c+L+R} \times \mathbb{Z}_2
\]

• *exactly massless* Goldstone mode \( \phi \) ("phonon")
• **Dense quark matter in compact stars – CFL** (p. 3/3)

Large, but not asymptotically large, densities: “switch on” $m_s$

![Diagram showing hadronic matter, spin-1 LOFF, 2SC, curCFL-K^0, CFL-K^0, CFL, and μ]

• **kaon-condensed CFL (CFL-K^0):** $U(1)_S$ spontaneously broken


• however: $U(1)_S$ not exact (weak interactions)
  → small mass of Goldstone mode $m \sim 50$ keV $\ll T_c \sim 10$ MeV

  D. T. Son, hep-ph/0108260
Towards the hydrodynamics of CFL ...

Astrophysicist: How many fluid components does CFL have?

 Particle physicist: ???

Astrophysicist: Is CFL a superfluid?

 Particle physicist: Yes, CFL breaks $U(1)_B$.

Astrophysicist: ???

 Particle physicist: $\text{CFL-K}^0$ also breaks $U(1)_S$, but that’s only an approximate symmetry.

Astrophysicist: ???
• Two-fluid picture of a superfluid (Helium-4) (page 1/2)

London, Tisza (1938); Landau (1941)
relativistic: Khalatnikov, Lebedev (1982); Carter (1989)

• “superfluid component”: condensate, carries no entropy

• “normal component”: excitations (Goldstone mode), carries entropy

• Hydrodynamic eqs.
  \[ \frac{\partial^2 \rho}{\partial t^2} = \Delta P \]
  \[ \frac{\partial^2 S}{\partial t^2} = \frac{S^2 \rho_s}{\rho_n} \Delta T \]
  \[ \Rightarrow \text{two wave eqs.} \]

  \[ u_1 = \sqrt{\frac{\partial P}{\partial \rho}}, \quad u_2 = \sqrt{\frac{s^2T\rho_s}{\rho c_V \rho_n}} \]
  \[ \Rightarrow \text{two sound velocities:} \]
• **Two-fluid picture of a superfluid (Helium-4) (page 2/2)**

• 1st sound: total density oscillates

• 2nd sound: relative densities of **superfluid** and **normal** components oscillate

• sound velocities of $^4\text{He}$

  ![Sound Velocities Graph](image)

  E. Taylor *et al.*, PRA 80, 053601 (2009)
  according to K.R. Atkins *et al.* (1953);
  V.P. Peshkov (1960)

→ How does the **two-fluid picture** arise from a **microscopic theory**?
• **Bose condensation and superfluid velocity (page 1/2)**

• start with simplest case:
  \( \phi^4 \) model

  \( \rightarrow \) from chiral Lagrangian for CFL mesons

  Bedaque, Schäfer, NPA 697, 802 (2002);

\[
\mathcal{L} = (\partial \phi)^2 - m^2|\phi|^2 - \lambda|\phi|^4
\]

\[
m^2 = m^2_{K^0} = \frac{m^2_s - m^2_d}{2\mu}
\]

\[
\lambda = \frac{4\mu^2_{K^0} - m^2_{K^0}}{6f^2_\pi}
\]

• \( \phi \rightarrow \phi + \phi \), condensate \( \phi = \frac{\rho}{\sqrt{2}}e^{i\psi} \)

• first step: no fluctuations \( (T = 0) \)

• minimize \( V(\rho) = -\mathcal{L} \)

\[
\rho^2 = \frac{(\partial \psi)^2 - m^2}{\lambda}
\]  

(assumption: \( \rho, \partial \psi \) const.)
• Bose condensation and superfluid velocity (page 2/2)

• “translation” at zero temperature (single fluid!) \((m = 0)\)

<table>
<thead>
<tr>
<th>Field-theoretically</th>
<th>Hydrodynamically</th>
</tr>
</thead>
<tbody>
<tr>
<td>(j^\mu)</td>
<td>(\frac{(\partial \psi)^2}{\lambda} \partial^\mu \psi)</td>
</tr>
<tr>
<td>(T^{\mu\nu})</td>
<td>(\frac{(\partial \psi)^2}{\lambda} \partial^\mu \psi \partial^\nu \psi - g^{\mu\nu} L)</td>
</tr>
</tbody>
</table>

• With \(\epsilon + P = \mu n\):

\[
P = \frac{(\partial \psi)^4}{4\lambda}, \quad \epsilon = \frac{3(\partial \psi)^4}{4\lambda}
\]

\[
\mu = |\partial \psi|, \quad n = \frac{|\partial \psi|^3}{\lambda}
\]

• superfluid velocity

\[
v^\mu = \frac{\partial^\mu \psi}{\mu}
\]

\(\Rightarrow\) irrotationality of superfluid, \(\nabla \times \vec{v} = 0\)
• From one fluid \((T = 0)\) to two fluids \((T > 0)\)

• qualitative change:
  – one fluid: \(\exists\) frame in which pressure is isotropic
  – two fluids: pressure anisotropic \(\forall\) frames

• formulation in terms of superfluid and normal fluid:

\[
j^{\mu} = n_s v^{\mu} + n_n u^{\mu}
\]

\[
T^{\mu\nu} = (\epsilon_s + P_s) v^{\mu} v^{\nu} - g^{\mu\nu} P_s + (\epsilon_n + P_n) u^{\mu} u^{\nu} - g^{\mu\nu} P_n
\]


• formulation in terms of entropy current and conserved current:

I.M. Khalatnikov and V.V. Lebedev, Phys. Lett. 91A, 70 (1982)

B. Carter and I. M. Khalatnikov, PRD 45, 4536 (1992)
• **Microscopic calculation at nonzero $T$ (page 1/2)**

- calculation for all $T \leq T_c$ needs self-consistent formalism;
  
  \[ 2\Pi (\nabla \psi = 0): \text{M. G. Alford, M. Braby, A. Schmitt, JPG 35, 025002 (2008)} \]
  
  \[ 2\Pi (\nabla \psi \neq 0): \text{M. G. Alford, S.K. Mallavarapu, S. Stetina, A. Schmitt, in preparation} \]

- here: one-loop (small $T$) effective action

\[
\frac{T}{V} \Gamma_{\text{eff}} = \frac{(\partial \psi)^4}{4\lambda} - \frac{1}{2V} \sum_k \text{Tr} \ln \frac{S^{-1}(k)}{T^2}
\]

- inverse tree-level propagator (at the $T = 0$ stationary point)

\[
S^{-1}(k) = \begin{pmatrix}
-k^2 + 2(\partial \psi)^2 & 2ik \cdot \partial \psi \\
-2ik \cdot \partial \psi & -k^2
\end{pmatrix}
\]

- anisotropic phonon dispersion ($\rightarrow$ first sound)

\[
\epsilon(\theta, k) = \frac{f(\theta)}{\sqrt{3}} k + \ldots, \quad f(\theta) = \frac{\sqrt{1 - \frac{v_s^2}{3}} \sqrt{1 - \frac{v_s^2}{3} (1 + 2 \cos^2 \theta)} + \frac{2|v_s|}{\sqrt{3}} \cos \theta}{1 - \frac{v_s^2}{3}}
\]
• **Microscopic calculation at nonzero $T$ (page 2/2)**

• compute current and stress-energy tensor

\[
j^\mu = \frac{\sigma^2}{\lambda} \partial^\mu \psi - \frac{1}{2V} T \sum_k \text{Tr} \left[ S \frac{\partial S^{-1}}{\partial \partial^\mu \psi} \right]
\]

\[
T^{\mu\nu} = \frac{(\partial \psi)^2}{\lambda} \partial^\mu \psi \partial^\nu \psi - g^{\mu\nu} \frac{(\partial \psi)^4}{4\lambda} - \frac{T}{V} \sum_k \text{Tr} \left[ S \frac{\partial S^{-1}}{\partial g^{\mu\nu} - u^\mu u^\nu} \right]
\]

[where $u^\mu = (1, 0, 0, 0)$]

• can be evaluated analytically for small $T$ (and all $v_s$), e.g.,

\[
T^{00} = \frac{\mu^4}{4\lambda} (1 - v_s^2)(3 + v_s^2) + \frac{\pi^2 T^4}{105 \sqrt{3} \mu^2 (1 - 3v_s^2)^3} (3 - 20v_s^2 + 9v_s^4)
\]

\[
- \frac{4\pi^2 T^6}{105 \sqrt{3} \mu^2 (1 - 3v_s^2)^6} (15 - 160v_s^2 - 774v_s^4 + 432v_s^6 + 135v_s^8) + \ldots
\]
• Relativistic two-fluid formalism (page 1/2)

• write stress-energy tensor as

\[ T^{\mu\nu} = -g^{\mu\nu}\Psi + j^{\mu}\partial^{\nu}\psi + s^{\mu}\Theta^{\nu} \]

• “generalized pressure” \( \Psi \):
  
  – \( \Psi \) is transverse pressure in “superfluid” and “normal” rest frames
  – \( \Psi \) depends on “momenta” \( \partial^{\mu}\psi, \Theta^{\mu} \)

\[ \Psi = \Psi[(\partial\psi)^2, \Theta^2, \partial\psi \cdot \Theta] \]

• “generalized energy density” \( \Lambda \equiv -\Psi + j \cdot \partial\psi + s \cdot \Theta \)
  
  – \( \Lambda \) is Legendre transform of \( \Psi \),
  – \( \Lambda \) depends on currents \( j^{\mu}, s^{\mu} \)

\[ \Lambda = \Lambda[j^2, s^2, j \cdot s] \]
- Relativistic two-fluid formalism (page 2/2)

\[ j^\mu = \frac{\partial \Psi}{\partial (\partial_\mu \Theta)} = B \partial^\mu \psi + A \Theta^\mu \]

\[ s^\mu = \frac{\partial \Psi}{\partial \Theta_\mu} = A \partial^\mu \psi + C \Theta^\mu \]

- conservation equations \( \partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0 \) become

\[ \partial_\mu j^\mu = 0, \quad \partial_\mu s^\mu = 0, \quad s_\mu \left( \partial^\mu \Theta^\nu - \partial^\nu \Theta^\mu \right) = 0 \]

“vorticity”

- in “mixed” form, we recover stress-energy tensor from


\[ T^{\mu\nu} = -g^{\mu\nu} \Psi + \frac{BC - A^2}{C} \partial^\mu \psi \partial^\nu \psi + \frac{1}{C} s^\mu s^\nu \]
• **Connect microscopic calculation with hydro (page 1/2)**

• microscopic calculation done in “normal rest frame” \( s^\mu = (s^0, 0, 0, 0) \)

• one can then show that

\[
\frac{T}{V} \Gamma_{\text{eff}} = \Psi
\]

• 8 independent degrees of freedom from 16 \((\partial^\mu \psi, \Theta^\mu, j^\mu, s^\mu)\)

\[
(\mu, \mu v^i_s, T) = (\partial^0 \psi, \partial^i \psi, \Theta^0) + \text{constraint } s^i = 0
\]

• obtain current \( j^\mu \) and entropy \( s^0 \) microscopically

• determine \( A, B, C \), (and \( \Theta^i \)), for instance

\[
A = \frac{s^0}{\partial^0 \psi} \left[ j^0 - \frac{j \cdot \nabla \psi}{(\nabla \psi)^2} \partial^0 \psi \right] \left[ j^0 - \frac{j \cdot \nabla \psi}{(\nabla \psi)^2} \partial^0 \psi + s^0 \Theta^0 \right]^{-1}
\]

etc.
• Connect microscopic calculation with hydro (page 2/2)

• use results to express $T$, $\mu$, $v_s$ in terms of Lorentz scalars $\sigma^2$, $\Theta^2$, $\partial \psi \cdot \Theta$

$\Rightarrow$ generalized pressure:

$$\Psi(\sigma^2, \Theta^2, \Theta \cdot \partial \psi) \simeq \frac{\sigma^4}{4\lambda} + \frac{\pi^2}{90\sqrt{3}} \left[ \Theta^2 + 2 \frac{(\partial \psi \cdot \Theta)^2}{\sigma^2} \right]^2 + \ldots$$

• “sonic metric” $G^{\mu\nu} \equiv g^{\mu\nu} + 2v_\mu v^\nu$ for $T^4$ term (linear part of Goldstone dispersion)

M. Mannarelli and C. Manuel, PRD 77, 103014 (2008)
• **Compute properties of the superfluid (page 1/2)**

• superfluid and normal charge densities (measured in normal frame)

\[ n_s = \frac{\mu^3}{\lambda}(1 - v_s^2) - \frac{4\pi^2T^4}{5\sqrt{3} \mu} \frac{1 - v_s^2}{(1 - 3v_s^2)^3}\]
\[ + \frac{8\pi^4T^6}{105\sqrt{3} \mu^3} \frac{1 - v_s^2}{(1 - 3v_s^2)^6}(95 + 243v_s^2 - 135v_s^4 - 27v_s^6) \]

\[ n_n = \frac{4\pi^2T^4}{5\sqrt{3} \mu} \frac{(1 - v_s^2)^2}{(1 - 3v_s^2)^3} - \frac{16\pi^4T^6}{35\sqrt{3} \mu^3} \frac{(1 - v_s^2)^2}{(1 - 3v_s^2)^6}(15 + 38v_s^2 - 9v_s^4) \]

(effect exaggerated by choosing \( \lambda \) very large)

• one fluid gets converted into the other by heating
• **Compute properties of the superfluid (page 2/2)**

• **sound velocities (measured in normal rest frame)**

\[
\begin{align*}
    u_1 &= \frac{\sqrt{3 - v_s^2(1 + 2\cos^2\theta)} \sqrt{1 - v_s^2} + 2|v_s|\cos\theta}{3 - v_s^2} + O(T^4) \\
    u_2 &= \frac{\sqrt{9(1 - v_s^2)(1 - 3v_s^2) + v_s^2\cos^2\theta + |v_s|\cos\theta}}{9(1 - v_s^2)} + \left(\frac{\pi T}{\mu}\right)^2 f(v_s^2, \cos\theta) + O(T^4)
\end{align*}
\]
Summary

- The hydrodynamics of CFL is nontrivial ...
  ... and poses fundamental questions regarding relativistic superfluid hydrodynamics and its microscopic, field-theoretical description.

- For the case of a $\varphi^4$ model we have connected the microscopic theory (at finite $T$) with the two-fluid formalisms of Son and Khalatnikov/Lebedev
• **Outlook**

• go beyond small-$T$ expansion  
  – solve stationarity eqs with superflow numerically  
  – compute superfluid density etc for all $T < T_c$

• how does the picture change with approximate (not exact) $U(1)_S$ symmetry? is superfluidity lost completely?  
  D. Parganlija, A. Schmitt, in preparation

• start from fermionic microscopic theory to account for $U(1)_B$

• put all this together for hydrodynamics of CFL-$K^0$

• include dissipation & non-uniform superflow