Shear viscosity of hadrons with K-Matrix cross sections
USTC Nuclear Theory Seminar

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with V. Koch, X.N. Wang & M. Prakash

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Motivation

The importance of $\eta_s/s$ in relativistic heavy-ions collisions

Methods to calculate transport coefficients
- The Chapman-Enskog approximation
- The relaxation time approximation
- The Green-Kubo method

Interactions
- Bertsch parameterization
- Breit-Wigner parameterization
- K-Matrix cross section
- Cross-section comparisons

Thermodynamics of interacting system

Results

Bulk Viscosity

Summary
Motivation

- Transport coefficients (shear & bulk viscosities) are important inputs to viscous hydrodynamics simulations of relativistic heavy ion collisions.
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- The present of shear viscosity will affect the magnitude of the elliptic flow.

- Comparison of different scheme of approximations in the calculation of transport coefficients.

- How jet transport coefficient $\hat{q}$ related to $\eta/s$.

- All of the above for bulk viscosity.
The ratio of $\eta/s$ and elliptic flow

The collective flow with viscous effects reduces the magnitude of elliptic flow relative to ideal (non-viscous) hydrodynamic motion.

Qualitative aspects of viscosity I

Shear Viscosity

\[ F = -\eta_s \frac{dv_x}{dy} \]  \hspace{1cm} (1)

Bulk Viscosity

\[ F = -\eta_v \nabla \cdot v \]  \hspace{1cm} (2)

F. Reif, Fundamental of statistical and Thermal Physics
Medium reaction to the external shear force.

Standard unit of viscosity is:

- SI unit: Pa.s (Pascal second) = $1 \text{ kg m}^{-1} \text{ s}^{-1}$
- cgs unit: P (Poise) = $1 \text{ gr cm}^{-1} \text{ s}^{-1}$

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The viscosity of water is $10^{-3}$ Pa.s at 20° C

The viscosity of RHIC matter is about $10^{11}$ Pa.s at 100 MeV ($\approx 10^{12}$ °C)

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The viscosity of water is \(10^{-3}\) Pa.s at \(20^{\circ}C\) \(^1\)

The viscosity of RHIC matter is about \(10^{11}\) Pa.s at 100 MeV
(\(\approx 10^{12}\) \(^{\circ}C\)) \(^2\)

Considering only the size of each system:

\[
\left( \frac{r_{\text{hadron}}}{r_{\text{molecule}}} \right)^2 \approx 10^{10}
\]

Temperature effect is also important.

\(^1\)Sears, Zemansky & Young, University Physics, 6th Edition

\(^2\)Wiranata & Prakash, 2009
Temperature dependent of $\eta$

- Estimation from kinetic theory:

$$\eta_s \approx \# \frac{\bar{p}}{\bar{\sigma}_T}, \quad \bar{\sigma}_T = \frac{1}{3} \int_{0}^{\infty} dg \, g^7 \, e^{-g^2} \int_{0}^{\pi} d\theta \, \sin^2 \theta \, \sigma \left( \frac{2g}{\sqrt{m\beta}}, \theta \right)$$

(4)

- $\bar{p}$: Mean momentum, $g$: Scaled (w.r.t. $\bar{p}$) relative momentum.
- $\bar{\sigma}_t(T)$: $T$-dependent transport cross section.
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- Relativistic expression are more complicated, but have similar content.

$$\eta \approx \# \frac{\bar{p}}{\bar{\sigma}_t(T)} = \# \frac{\hbar}{\lambda \bar{\sigma}_t(T)} := \text{action} \div \text{physical volume}$$
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$$\eta \approx \# \frac{\bar{p}}{\bar{\sigma}_T(T)} = \# \frac{\hbar}{\lambda \bar{\sigma}_T(T)} := \frac{\text{action}}{\text{physical volume}}$$ \hspace{1cm} (5)

- Above, $\lambda$ is the thermal de-Broglie wavelength.

$$\lambda \ & \ \eta \propto \left\{ \begin{array}{l}
\frac{1}{\sqrt{mT}} \\
\frac{\sqrt{mT}}{\bar{\sigma}_T(T)}
\end{array} \right\} \begin{array}{l}
\&
\frac{1}{T} \\
\& \frac{T}{\bar{\sigma}_T(T)}
\end{array} \begin{array}{l}
\text{for NR}
\text{for UR}
\end{array}$$ \hspace{1cm} (6)
Fluidity vs the applicability of hydrodynamics

- The applicability of hydrodynamics requires that the effective mean free path to be much smaller compared to the size of the system.

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3 J. Liao & V. Koch, 2010
The applicability of hydrodynamics requires that the effective mean free path to be much smaller compare to the size of the system.

The concept of fluidity of a substance can only be meaningful if the fluidity is defined exclusively in terms of properties of the substance itself. \(^3\) Mean free path vs inter particle distance (density-density correlation length).

\(^3\) J. Liao & V. Koch, 2010
The ratio of $\eta/s$

From Bjorken expansion (1+1D)

$$\frac{de}{d\tau} = - \frac{e + P - 4\eta_s/(3\tau)}{\tau} = - \frac{sT}{\tau} \left( 1 - \frac{4}{3} \frac{\eta_s}{s} \frac{1}{T\tau} \right),$$  \hspace{1cm} (7)
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Enthalpy is given as follow:

$$w_R = \epsilon + p = Ts + \mu_R n; \quad \mu_R = \mu_{NR} + m;$$

In general

$$w = Ts + \mu n + mn; \tag{8}$$

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4Landau, Fluid Mechanics
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The Kinematic viscosity reads as

$$
\frac{\eta_s}{s} (\text{Relativistic}) \approx \frac{\eta_s}{\rho} (\text{Non-Relativistic}) \approx \frac{\eta_s}{w} (\text{General})
$$

(9)

\footnote{Landau, Fluid Mechanics}
Methods to calculate transport coefficients

In Hadronic system at Relativistic heavy-ions collisions, here three available approximations/methods to calculate transport coefficients:

1. The Chapman-Enskog approximation

2. The relaxation time approximation

The Chapman-Enskog approximation I

“For small deviation from equilibrium, the distribution function can be expressed in term of hydrodynamics variables and their gradients. Transport coefficients such as shear and bulk viscosities are calculable from kinetic theory.”

\(^5\) van Leeuween
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\[ f(x, p) = f^0 [1 + \phi(x, p)]; \quad \phi(x, p) = \text{deviation function} \quad (10) \]
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\[ f^0(x, p) = \frac{1}{(2\pi \hbar)^3} \exp \left[ \frac{m u(x) + p_\alpha u^\alpha(x)}{kT} \right] \quad \text{equilibrium dist. function} \quad (11) \]

where

\( \mu(x) \) is chemical potential,

\( u(x) \) is flow velocity and

\( T \) is temperature.

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van Leeuwen
The general form of the deviation function:

\[ \phi(x, p) = A \partial_\alpha U^\alpha \quad \text{(Bulk Viscosity)} \]
\[ + B \Delta_{\alpha\beta} p^\beta \Delta^{\alpha\beta} (T \partial_\beta T + C^{-2} DU_\beta) \quad \text{(Heat Conductivity)} \]
\[ + C \langle p_\alpha p_\beta \rangle \langle \partial^\alpha U_\beta \rangle \quad \text{(Shear Viscosity)} \] (12)

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\(^{6}\)Details can be found in A. Wiranata, PRC 2012
The Chapman-Enskog approximation II

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By using kinetic theory

\[ T^{\alpha\beta} = c \int p^\alpha p^\beta f(x, p) \frac{d^3p}{p^0} \quad (13) \]

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and a little bit of math\(^6\)

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The Chapman-Enskog approximation III

The working expression

$$\eta_s = \frac{1}{10} (kT)^2 \sum_{m=0}^{r-1} c_m \gamma_m ; \quad \sum_{m=0}^{r-1} c_m^{(r)} c_{mn} = \frac{\gamma_n}{\rho kT} ; \quad (n = 0, 1, \cdots, r - 1) \quad (14)$$

$r$ is the order of approximation.
The Chapman-Enskog approximation III

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\( r \) is the order of approximation.

\[ [\eta_s]_1 = \frac{1}{10} kT \frac{\gamma_0^2}{c_{00}} \quad \text{1st order approximation} \] (15)

\[ [\eta_s]_2 = \frac{1}{10} kT \frac{\gamma_0^2 c_{11} - 2\gamma_0 \gamma_1 c_{01} + \gamma_1^2 c_{00}}{c_{00} c_{11} - c_{01}^2} \quad \text{2nd order} \] (16)
Shear Viscosity (the 1st order approximation)

\[
[\eta_s]_1 = \frac{1}{10} kT \frac{\gamma_0^2}{c_{00}}; \quad \gamma_0 = -10\hat{h}; \quad \hat{h} = \frac{K_3(z)}{K_2(z)},
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where \( z = m/T \) is relativity parameter.
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\[ c_{00} = 16 \left( w_2^{(2)} - \frac{1}{z} w_1^{(2)} + \frac{1}{3z^2} w_0^{(2)} \right); \quad w_i^{(s)} = \text{Rel. Omega Integral} \]  

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\]

Relativistic Omega Integral:

\[
w^{(s)}_i = \frac{2\pi z^3 c}{K_2(z)} \int_0^\infty d\psi \sinh^7 \psi \cosh \psi K_j (2z \cosh \psi) \\
\times \int_0^{\pi} d\theta \sin \theta \sigma(\psi, \theta) (1 - \cos^2 \theta),
\]  

where relative momentum \((g = mc \sinh \psi)\), total momentum \((P = 2mc \cosh \psi)\), differential cross section \((\sigma(\psi, \theta))\) and \(j = 5/2 + (-1)^i/2\).
Relaxation time approximation

The effect of collisions is always to bring perturbed distribution function close to equilibrium distribution function. In other words, the effect of collisions is to restore local equilibrium distribution function exponentially with relaxation time $\tau(p)$ which is in the order of time requires between collisions.

$$C[f] \approx \frac{f(x, p) - f_{0eq}(x, p)}{\tau(p)} = D_t f(x, p)$$

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8 Wiranata & M. Prakash, 2012
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Shear viscosity (for $a + b \rightarrow c + d$) is given by $^7$

$$\eta_s = \frac{1}{15T} \int_0^\infty \frac{d^3 p_a}{(2\pi)^3} \frac{|p_a|^4}{E_a^2} \frac{1}{w_a(E_a)} f^\text{eq}_a$$ (21)

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The collision frequency, $w_a(E_a)$ is given by $^8$

$$w_a(E_a) = \int_0^{\infty} \frac{d^3p_b}{(2\pi)^3} \frac{\sqrt{s(s - 4m^2)}}{2E_a2E_b} f^{eq}_b \sigma_T,$$  \hspace{1cm} (22)

where $\sigma_T$ is the total cross section.


$^8$Wiranata & M.Prakash, 2012
The Green-Kubo method enables us to extract transport coefficients in terms of the correlation functions of fluctuations in the stress-energy tensor for the system\(^9\). Shear viscosity is given by

\[
\eta_s = \frac{1}{T} \int d^3 r \int dt \langle T_{xy}(\vec{0}, 0) T_{xy}(\vec{r}, t) \rangle,
\]

(23)

where \(T\) is temperature, \(T_{xy}\) is the shear component of stress energy tensor. The averaging symbol refers to Gibbs grand canonical average.

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The stress-energy tensor is given by

\[ T^{\mu\nu}(\vec{r}, t) = \int d^3p \frac{p^{\mu}p^{\nu}}{p^0} f(\vec{r}, \vec{p}, t), \]  

(24)

where \( f(\vec{r}, \vec{p}, t) \) is the distribution function of the system.

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\(^{9}\text{S.G. Brush, Kinetic Theory, 1972}\)
The Green-Kubo methods II

In the case of point particles (monte-carlo simulations & lattice calculations), the distribution function and stress energy tensor of the system can be written as follow\(^{10}\):

\[
    f(\vec{r}, \vec{p}) = \frac{1}{V} \sum_{j=1}^{N_{\text{part}}} \delta^{(3)}(\vec{p} - \vec{p}_j), \quad T^{xy} = \frac{1}{V} \sum_{j=1}^{N_{\text{part}}} \frac{p_x(j)p_y(j)}{p^0(j)}
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\(^{10}\)Demir & Bass, PRL 2009
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\]

and then shear viscosity reads as

\[
\eta_s = \frac{T}{V} \int_{0}^{\infty} dt \langle T^{xy}(0) T^{xy}(t) \rangle \tag{26}
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(26)

Requires knowledge of all components of the stress energy-momentum tensor for all times. This entails dynamical simulations, often termed as molecular dynamical calculations with specified force laws and equation of motions.

\(^{10}\)Demir & Bass, PRL 2009
Interactions is entering shear viscosity calculation in the scattering cross section \( \sigma(s, \theta) \).
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Shear Viscosity:

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$$\eta_s = \frac{1}{15T} \int_0^\infty \frac{d^3p_a}{(2\pi)^3} \frac{|p_a|^4}{E_a^2} \frac{1}{w_a(E_a)} f_{eq}^a , \quad \text{total cross section}$$

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\eta_s = \frac{T}{V} \int_0^\infty dt \left\langle T^{xy}(0) T^{xy}(t) \right\rangle, \quad \text{total cross section}
\]

Cross-Sections are:

3. K-Matrix formalism (This work).
Bertsch parameterization

Diff. Cross-section \((\pi - \pi \rightarrow M_R \rightarrow \pi - \pi)\):

\[
\sigma = \frac{4}{q_{cm}^2} \left( \frac{1}{9} \sin^2 \delta_0^0 + \frac{5}{9} \sin^2 \delta_0^2 + \frac{1}{3} \sin^2 \delta_1^1 \right)
\]

Phase-shifts:

\[
\delta_0^0(\epsilon) = \frac{\pi}{2} + \arctan \left( \frac{\epsilon - m_\sigma}{\Gamma_\sigma/2} \right)
\]

\[
\delta_1^1(\epsilon) = \frac{\pi}{2} + \arctan \left( \frac{\epsilon - m_\rho}{\Gamma_\rho/2} \right)
\]

\[
\delta_0^2(\epsilon) = -\frac{0.12q}{m_\pi}
\]

Decay widths:

\[
\Gamma_\rho = 0.095q \left( \frac{q/m_\pi}{1 + (q/m_\rho)^2} \right)^2
\]

\[
\Gamma_\sigma = 2.06q
\]

Breit-Wigner parameterization

Tot. Cross-section (mesons-baryons):

$$\sigma_t = \sum_{\text{Res}} \langle j_B, m_B, j_M, m_M | j_R, m_R \rangle \frac{2S_R + 1}{(2S_B + 1)(2S_m + 1)} \times \frac{\pi}{p_{\text{cms}}^2} \frac{\Gamma_{R \rightarrow M, B} \Gamma_{\text{tot}}}{(M_R - \sqrt{s})^2 + \frac{\Gamma_{\text{tot}}^2}{4}},$$

Decay widths:

$$\Gamma_{i,j}(\sqrt{s}) = \Gamma_R^{i,j} \frac{M_R}{\sqrt{s}} \left( \frac{p_{i,j}(\sqrt{s})}{p_{i,j}(M_R)} \right)^{2l+1} \times \frac{1.2}{1 + 0.2 \left( \frac{p_{i,j}(\sqrt{s})}{p_{i,j}(M_R)} \right)^{2l}},$$

where $M_R$ is the resonance mass, $\Gamma_R^{i,j}$ is decay width at the pole and $l$ is the angular momentum of the exit channel.

Parameterization from Bass et.al, 1999
This formalism was introduced by Wigner and Eisenbud for the study of resonances in nuclear reactions in order to maintain the unitarity of the T-matrix\textsuperscript{11}

\textsuperscript{11}Wigner, 1946 & 1947
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The scattering operator and can be decomposed into

\[ S = I + 2iT; \quad I(\text{identity operator}); \quad T - \text{matrix(Interaction terms)} \]  

\textsuperscript{11}Wigner, 1946&1947
K-Matrix cross section I (Why another cross section ??)

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Introducing the \( K \) operator as follow:

\[ K = T^{-1} + iI, \quad K = K^\dagger. \]  \hspace{1cm} (28)

The K-matrix is symmetric and real.

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K = T^{-1} + iI, \quad K = K^\dagger.
\] (28)

The K-matrix is symmetric and real.

We can write $T$ matrix into its real and imaginary parts as follow
\[
\begin{align*}
\text{Re} T &= (I + K^2)^{-1}K = K(I + K^2)^{-1} \\
\text{Im} T &= (I + K^2)^{-1}K^2 = K^2(I + K^2)^{-1}, \quad T = e^{i\delta} \sin \delta
\end{align*}
\] (29)

Details can be found in A.Wiranata et.al (In progress)

\(^{11}\)Wigner, 1946&1947
In the case of resonances system, $K$–matrix can be written as follow

$$K_{ij} = \sum_{\alpha} \frac{m_{\alpha} \Gamma_{\alpha i}}{m_{\alpha}^2 - m^2}, \quad \alpha = \text{Resonances}$$

where the partial decay width is given as follow:

$$\Gamma_{\alpha i}(m) = \Gamma_0 \left( \frac{M_R}{m} \right) \left( \frac{q(m)}{q(m_R)} \right) (B_{\alpha i}^l(q, q_\alpha))^2$$

and the $b$-factor is defined as follow:

$$B_{\alpha i}^l(q, q_\alpha) = \frac{F_l(q)}{F_l(q_\alpha)} \quad (30)$$

details of $f$-factor can be found in A.Wiranata (in progress)
Single resonances

\[ \pi - \pi \rightarrow \rho \rightarrow \pi - \pi \]
Two and more resonances

\[ \pi - \pi \rightarrow M_{R1}, M_{R2} \rightarrow \pi - \pi \]
Two and more resonances

\[ \pi - \pi \rightarrow M_{R1}, M_{R2} \rightarrow \pi - \pi \]
Bertsch parameterization can NOT access higher resonances.
Take home points

1. Bertsch parameterization can NOT access higher resonances.
2. Breit-Wigner/UrQMD parameterization only “good” for a single resonance.
Take home points

1. Bertsch parameterization can NOT access higher resonances.
2. Breit-Wigner/UrQMD parameterization only “good” for a single resonance.
3. K-matrix can include higher resonances and maintain unitarity at the same time.
Methods to calculate equation of state

1. Non-interacting hadrons gas: Pressure, energy density and entropy density of the system are sum of each individual pressure, energy density and entropy density of each individual particles in the system.

12. Kahn & Uhlenbeck, 1938
13. R. Venugopalan & M. Prakash, 1992
Methods to calculate equation of state

1. Non-interacting hadrons gas: Pressure, energy density and entropy density of the system are sum of each individual pressure, energy density and entropy density of each individual particles in the system.

2. Cluster-expansion\(^{12}\): Pressure, energy density and entropy density of the system can be calculated from virial expansion\(^{13}\).

\(^{12}\)Kahn & Uhlenbeck, 1938
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Methods to calculate equation of state

1. Non-interacting hadrons gas: Pressure, energy density and entropy density of the system are sum of each individual pressure, energy density and entropy density of each individual particles in the system.

2. Cluster-expansion\(^{12}\): Pressure, energy density and entropy density of the system can be calculated from virial expansion\(^ {13}\).

\[ P = \frac{1}{\beta} \frac{1}{2\pi^3} \int_{2m_\pi}^{\infty} d\epsilon \epsilon^2 K_1(\epsilon\beta) \sum_{l,I} g_I^l \delta^I_l(\epsilon), \]  \hspace{1cm} (31)

\[ E = \frac{1}{4\pi^3} \int_{2m_\pi}^{\infty} d\epsilon \epsilon^3 [K_2(\epsilon\beta) + K_0(\epsilon\beta)] \sum_{l,I} g_I^l \delta^I_l(\epsilon), \]  \hspace{1cm} (32)

\[ s = \beta \frac{1}{2\pi^3} \int_{2m_\pi}^{\infty} d\epsilon \epsilon^3 K_2(\epsilon\beta) \sum_{l,I} g_I^l \delta^I_l(\epsilon). \]  \hspace{1cm} (33)

\(^{12}\)Kahn & Uhlenbeck, 1938

\(^{13}\)R.Venugopalan & M. Prakash, 1992
Results
Trace anomaly

\[ \pi - \pi \rightarrow \text{Resonances} \rightarrow \pi - \pi \]

3 quarks and \( m_s = 10m_{u,d} \), P. Petreczky (HotQCD Collaboration), 2010
The ratio of $\eta_s/s$

$\pi - \pi \rightarrow \rho \rightarrow \pi - \pi$

![Graph showing the ratio $\eta_s/s$ vs. temperature $T$ (MeV)]
Shear viscosity & entropy density

\[ \pi - \pi \rightarrow \text{Resonances} \rightarrow \pi - \pi \]

Shear viscosity

Entropy density

\[ \eta(T) \] \hspace{2cm} \[ S/T^3 \]
The ratio of $\eta_s/s$ II

$\pi - \pi \rightarrow \text{Resonances} \rightarrow \pi - \pi$

All Resonances included

![Graph 1](image1)

Single vs All Resonances

![Graph 2](image2)
Bulk Viscosity
The 1st approximation:

\[ \eta_v = \frac{9}{8} kT \frac{\left[ z \hat{h} (\gamma - 5/3) + \gamma \right]^2}{\omega_0^{(2)}} , \quad z = \frac{m}{T} , \quad \hat{h} = \frac{K_3(z)}{K_2(z)} , \quad \gamma \equiv \frac{c_p}{c_v} \]  (34)
Bulk viscosity (working expression)

The 1st approximation:

\[ \eta_v = \frac{9}{8} kT \left( z \hat{h}(\gamma - 5/3) + \gamma \right)^2, \quad z = \frac{m}{T}, \quad \hat{h} = \frac{K_3(z)}{K_2(z)}, \quad \gamma \equiv \frac{c_p}{c_v} \] (34)

Relativistic Omega Integral:

\[ w_i^{(s)} = \frac{2\pi z^3 c}{K_2(z)} \int_0^\infty d\psi \sinh^7 \psi \cosh \psi K_j(2z \cosh \psi) \]
\[ \times \int_0^\pi d\theta \sin \theta \sigma(\psi, \theta)(1 - \cos^2 \theta), \] (35)
Bulk viscosity (working expression)

The 1st approximation:

\[ \eta_v = \frac{9}{8} kT \frac{\left[ z \hat{h}(\gamma - 5/3) + \gamma \right]^2}{w_0^{(2)}} , \quad z = \frac{m}{T} , \quad \hat{h} = \frac{K_3(z)}{K_2(z)} , \quad \gamma \equiv \frac{c_p}{c_v} \]  \hspace{1cm} (34)

Relativistic Omega Integral:

\[ w_i^{(s)} = \frac{2\pi z^3 c}{K_2(z)} \int_0^\infty d\psi \sinh^7 \psi \cosh \psi K_j(2z \cosh \psi) \]
\[ \times \int_0^\pi d\theta \sin \theta \sigma(\psi, \theta)(1 - \cos^2 \theta) , \]  \hspace{1cm} (35)

where relative momentum \( (g = mc \sinh \psi) \), total momentum \( (P = 2mc \cosh \psi) \), differential cross section \( (\sigma(\psi, \theta)) \) and \( j = 5/2 + (-1)^i/2 \).
Bulk Viscosity of Pion Gas

Constant cross section $\sigma = 10$ mb.

$\pi - \pi \rightarrow \text{Resonances} \rightarrow \pi - \pi$

A. Wiranata & M. Prakash, 2008
Bulk viscosity & speed of sound

1st approximation:

\[ \eta_v = kT \frac{a^2 \left( \frac{1}{3} - v_s^2 \right)^2 + 2ab \left( \frac{1}{3} - v_s^2 \right) + b^2}{2 \omega_0^{(2)}} \]

where \( z = m/T \), \( a = -3/2(z\hat{h} + 1) \), \( b = -1/2(z\hat{h} - 4) \) and \( \hat{h} = K_3(z)/K_2(z) \).

- Nearly massless particle or very high temperature \( z \ll 1 \).
  as \( z \to 0; z\hat{h} \to 4, a \to -\frac{15}{2} \) and \( b \to 0 \)

\[ \eta_v = \frac{225}{8} \frac{kT}{w_0^{(2)}} \left( \frac{1}{3} - v_s^2 \right)^2 \quad v_s^2 = 1/3 \to \eta_s = 0 \]
Bulk viscosity & speed of sound

1st approximation:

\[
\eta_v = kT \frac{a^2 (1/3 - v_s^2)^2 + 2ab (1/3 - v_s^2) + b^2}{2\omega_0^{(2)}},
\]

where \( z = m/T, \quad a = -3/2(z\hat{h} + 1), \quad b = -1/2(z\hat{h} - 4) \) and \( \hat{h} = K_3(z)/K_2(z). \)

1. Nearly massless particle or very high temperature \( z \ll 1 \).
   As \( z \to 0; \ z\hat{h} \to 4, \ a \to -\frac{15}{2} \) and \( b \to 0 \)

\[
\eta_v = \frac{225}{8} \frac{kT}{w_0^{(2)}} \left( \frac{1}{3} - v_s^2 \right)^2 \quad v_s^2 = 1/3 \to \eta_s = 0
\]

2. Massive particle or very low temperature \( z \gg 1 \).
   As \( a \to -\frac{3}{2}z \) and \( b \to -1/2z \)

\[
\eta_v = \frac{225}{8} \frac{kT}{w_0^{(2)}} \left( \frac{1}{3} - v_s^2 \right)^2 \quad v_s^2 = 1/3 \to \eta_s = 0
\]
Bulk viscosity & speed of sound

2nd and higher Approximations:

\[ [\eta_v]_2 = \frac{\rho k T}{m} (a_2 \alpha_2 + a_3 \alpha_3) , \quad [\eta_v]_3 = \frac{\rho k T}{m} (a_2 \alpha_2 + a_3 \alpha_3 + a_4 \alpha_4) , \quad (39) \]

where \( a_2, a_2, a_2 \) contain interaction terms (omega integrals) & \( \alpha_2, \alpha_3, \alpha_4 \) feature thermodynamic variables.
Bulk viscosity & speed of sound

2nd and higher Approximations:

\[
[\eta_v]_2 = \frac{\rho k T}{m} (a_2 \alpha_2 + a_3 \alpha_3), \quad [\eta_v]_3 = \frac{\rho k T}{m} (a_2 \alpha_2 + a_3 \alpha_3 + a_4 \alpha_4),
\]  

(39)

where \( a_2, a_2, a_2 \) contain interaction terms (omega integrals) & \( \alpha_2, \alpha_3, \alpha_4 \) feature thermodynamic variables.

\[
\alpha_3 = \hat{h} \left[ -\frac{1}{2} z^2 - \frac{11}{2} - \left( \frac{3}{2} z^2 + \frac{1}{4} z \right) \left( \frac{1}{3} - v_s^2 \right) \right] + \frac{1}{6} z^2 + 2z + \frac{17}{3}
\]  

(40)

\[
\alpha_4 = \hat{h} \left[ \left( -\frac{1}{3} z^3 - \frac{17}{21} z^2 - \frac{23}{16} z \right) - \left( z^3 + \frac{7}{4} z^2 - \frac{9}{16} z \right) \left( \frac{1}{3} - v_s^2 \right) \right] + \frac{1}{6} z^3 + \frac{5}{3} z^2 + \frac{17}{3} z + \frac{23}{4}
\]  

(41)

Terms that feature speed of sound dependences.
Bulk viscosity & speed of sound

Analysis for limiting cases:

1. Nearly massless particles \((z = m/T \rightarrow 0, \hat{z}h \rightarrow 4)\)

\[
\alpha_3 \rightarrow -\frac{11}{12} \hat{z}h - \frac{1}{4} \hat{z}h \left(\frac{1}{3} - v_s^2\right) + \frac{11}{3} - \frac{11}{4} \left(\frac{1}{3} - v_s^2\right) \tag{42}
\]

\[
\alpha_4 \rightarrow -\frac{23}{16} \hat{z}h - \frac{9}{16} \hat{z}h \left(\frac{1}{3} - v_s^2\right) + \frac{23}{4} - \frac{69}{16} \left(\frac{1}{3} - v_s^2\right) \tag{43}
\]

For \(v_s^2 \rightarrow \frac{1}{3}\) then \(\alpha_3, \alpha_4 = 0\)
Bulk viscosity & speed of sound

Analysis for limiting cases:

1. Nearly massless particles \((z = m/T \to 0, \, z\hat{h} \to 4)\)

\[
\alpha_3 \to -\frac{11}{12} z\hat{h} - \frac{1}{4} z\hat{h} \left( \frac{1}{3} - v_s^2 \right) + \frac{11}{3} - \frac{11}{4} \left( \frac{1}{3} - v_s^2 \right) \tag{42}
\]

\[
\alpha_4 \to -\frac{23}{16} z\hat{h} - \frac{9}{16} z\hat{h} \left( \frac{1}{3} - v_s^2 \right) + \frac{23}{4} - \frac{69}{16} \left( \frac{1}{3} - v_s^2 \right) \tag{43}
\]

For \(v_s^2 \to \frac{1}{3}\) then \(\alpha_3, \alpha_4 = 0\)

2. Massive particles \((z = m/T \gg 1)\)

\[
\alpha_3 \to \hat{h} \left[ -\frac{1}{2} z^2 - \frac{3}{2} z^2 \left( \frac{1}{3} - v_s^2 \right) \right] - \frac{1}{6} z^2 - \frac{1}{2} z^2 \left( \frac{1}{3} - v_s^2 \right) \tag{44}
\]

\[
\alpha_4 \to \hat{h} \left[ -\frac{1}{3} z^3 - z^3 \left( \frac{1}{3} - v_s^2 \right) \right] - \frac{1}{6} z^3 + \frac{1}{2} z^3 \left( \frac{1}{3} - v_s^2 \right) \tag{45}
\]

For \(v_s^2 \to \frac{2}{3}\) then \(\alpha_3, \alpha_4 = 0\)

For a given \(T\), intermediate relativity mass particles contribute significantly to \(\eta_v\).

A. Wiranata & M. Prakash, 2008
Bulk viscosity & Inelastic Collisions

Internal excitation and creation of new species of particles contribute to bulk viscosity

\[
\eta_v = \left( \frac{c_{\text{int}}}{c_v} \right)^2 kT \frac{1}{\gamma_{tt}},
\]

where \( c_{\text{int}} \) & \( c_v \) are the internal heat capacity & heat capacity at fixed volume per molecule and

\[
\gamma_{tt} = 2\sqrt{\frac{kT}{\pi m}} \left( \sum_i e^{-\epsilon_i} \right)^{-2} \sum_{ijkl} e^{-\epsilon_i-\epsilon_j} \times \int e^{-g^2} g^3 (\Delta \epsilon)^2 \sigma \sin \chi \, d\chi \, d\psi \, dg
\]

For \( \Delta \epsilon \to 0 \), then \( \eta_v \to 0 \).

In the non-relativistic limit \( (z = m/T \gg 1) \), inelastic part of the cross section contribute the most to bulk viscosity. \text{Wang et al., 1964}
Bulk Viscosity of Pion Gas

\[ \eta_v = \left[ F_0^8 / (m_\pi c^2)^5 \right] \exp\left(2m_\pi c^2 / k_B T\right), \text{ where } F_0 \approx 93 \text{ MeV.} \]

\[ \pi + \pi \leftrightarrow \pi + \pi + \pi + \pi + \pi \]

Figure 4.3: Left panel: Bulk viscosity versus temperature from the number changing inelastic process \( \pi + \pi \leftrightarrow \pi + \pi + \pi + \pi + \pi \). Right panel: The ratio of bulk viscosity to entropy density versus \( z^{-1} = k_B T / m_\pi c^2 \). This figure is taken from Lu and Moore, Phys. Rev. C 83 (2011) 044901.
Bulk Viscosity (Preliminary results)

\[ \pi - \pi \rightarrow \text{Resonances} \rightarrow \pi - \pi \]
Summaries:

- It is important to include higher resonances in the shear viscosity and entropy density calculation, particularly in the regime near $T_c$. The magnitude of shear viscosity will be lower and it will be higher for entropy density.
- One needs to be careful when include more than one resonances in the interaction cross section (unitarity of the $T-$matrix needs to be conserved).

Outlooks:

- Multi-components system which consider Mesons-Baryons interactions.
- Hadronic energy loss.
- The ratio of bulk viscosity to entropy density.