\(D^0 - \overline{D}^0\) Mixing and CP Violation

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The oscillation in time of neutral \(D\) mesons into their antiparticles, and \textit{vice versa}, commonly called \(D^0 - \overline{D}^0\) mixing, was discovered by BABAR and Belle in 2007, and confirmed shortly afterwards by CDF. In these experiments the flavor of the neutral \(D\) meson is tagged by the charge of the slow pion from \(D^*\) decay, or is ignored. The formalism for mixing in these cases is that of an isolated meson. When a pair of neutral \(D\) mesons is produced at the \(\Psi(3770)\) the charge conjugation quantum number of the system is -1. The matrix elements for the time-dependent final states are antisymmetric under the interchange of the two \(D\) meson amplitudes, leading to correlations in rates related to mixing and CP violation. In this talk I will review basic ideas related to mixing, the state of the field today, and then focus on how well a charm threshold factory could measure mixing parameters using time-dependent, correlated, decay rates.
Neutral $D$ mesons are produced as flavor eigenstates $D^0$ and $\bar{D}^0$ and decay via
\[ i \frac{\partial}{\partial t} \begin{pmatrix} D^0(t) \\ \bar{D}^0(t) \end{pmatrix} = \left( M - \frac{i}{2} \Gamma \right) \begin{pmatrix} D^0(t) \\ \bar{D}^0(t) \end{pmatrix} \]
as mass and lifetime eigenstates $D_1, D_2$
\[
|D_1\rangle = p|D^0\rangle + q|\bar{D}^0\rangle \\
|D_2\rangle = p|D^0\rangle - q|\bar{D}^0\rangle \\
\]
where $|q|^2 + |p|^2 = 1$ and
\[
\begin{pmatrix} q \\ p \end{pmatrix}^2 = \frac{M_{12}^* - \frac{i}{2} \Gamma_{12}^*}{M_{12} - \frac{i}{2} \Gamma_{12}}
\]

$D_1, D_2$ have masses $M_1, M_2$ and widths $\Gamma_1, \Gamma_2$

Mixing occurs when there is a non-zero mass
\[ \Delta M = M_1 - M_2 \]
or lifetime difference
\[ \Delta \Gamma = \Gamma_1 - \Gamma_2 \]

For convenience define, $x$ and $y$
\[ x = \frac{\Delta M}{\Gamma}, \quad y = \frac{\Delta \Gamma}{2\Gamma} \]
where
\[ \Gamma = \frac{\Gamma_1 + \Gamma_2}{2} \]

And define the mixing rate
\[ R_M = \frac{x^2 + y^2}{2} \]
CP Violation Simplified

**CP violation in mixing** originates in the difference between mixing and CP eigenstates:

\[ |D_1\rangle = p|D^0\rangle + q|\bar{D}^0\rangle \]
\[ |D_2\rangle = p|D^0\rangle - q|\bar{D}^0\rangle \quad q/p \neq 1 \]

**Direct CP violation** originates in the difference between the magnitudes of CP-conjugate decays:

\[ |A(D \to f)| \neq |\bar{A}(D \to \bar{f})| \]

If direct CP violation is absent, or small, then the four observables in mixing-related CPV

\[ x, \ y, \ |q/p|, \ \arg(q/p) \]

are related to three underlying parameters

\[ x_{12} \equiv 2|M_{12}|/\Gamma, \ y_{12} \equiv |\Gamma_{12}|/\Gamma, \ \phi_{12} \equiv \arg(M_{12}/\Gamma_{12}) \]
How Mixing is Calculated

\[
\left(M - \frac{i}{2} \Gamma \right)_{12} = \frac{1}{2m_D} \langle D^0|\mathcal{H}^{\Delta C=2}_w|\bar{D}^0 \rangle \\
+ \frac{1}{2m_D} \sum_n \frac{\langle D^0|\mathcal{H}^{\Delta C=1}_w|n \rangle \langle n|\mathcal{H}^{\Delta C=1}_w|\bar{D}^0 \rangle}{m_D - E_n + i\epsilon}
\]

The first term is called the short distance contribution and the second the long distance contribution. Assuming the short distance contributions are small, and that CP is conserved, we can express \( y \) as the absorptive part of the second term

\[
y = \frac{1}{\Gamma_D} \sum_n \rho_n \langle \bar{D}^0|\mathcal{H}^{\Delta C=1}_w|n \rangle \langle n|\mathcal{H}^{\Delta C=1}_w|D^0 \rangle,
\]

where \( \rho_n \) is the phase space factor corresponding to the charmless intermediate state \(|n\rangle\).

Points of theoretical consensus

- Short distance contributions to \( x \) and \( y \) are \( \ll 10^{-2} \);
- CP is not significantly violated in the Standard Model;
- Large long-distance contributions to \( y \) may originate in the different phase spaces available for CP-even and CP-odd final states (but not in SM matrix elements); \( y \sim \mathcal{O}(10^{-2}) \) cannot be excluded in the Standard Model; \( x \sim \mathcal{O}(10^{-2}) \) is less likely, although it cannot be excluded absolutely.
- New Physics may contribute to mixing at the \( x, y \sim \mathcal{O}(10^{-2}) \) level.
Box diagram SM charm mixing rate
naively expected to be very low
\(R_M \sim 10^{-10}\) (Datta & Kumbhakar)

\[ Z.\text{Phys. C27, 515 (1985)} \]

CKM suppression \(\rightarrow |V_{ub}V_{cb}^*|^2\)
GIM suppression \(\rightarrow (m_s^2 - m_d^2)/m_W^2\)
Di-penguin mixing, \(R_M \sim 10^{-10}\)

\[ \text{Phys. Rev. D 56, 1685 (1997)} \]

Enhanced rate SM calculations
generally due to long-distance contributions:

first discussion, L. Wolfenstein

\[ \text{Phys. Lett. B 164, 170 (1985)} \]

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Standard Model Mixing Predictions

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Partial History of Long-Distance Calculations

- Early SM calculations indicated long distance contributions produce $x \ll 10^{-2}$:
  - $x \sim 10^{-3}$ (dispersive sector)
    - \text{PRD} 33, 179 (1986)
  - $x \sim 10^{-5}$ (HQET)
    - \text{Nucl. Phys.} B403, 605 (1993)

- More recent SM predictions can accommodate $x, y \sim 1\%$ [of opposite sign] (Falk et al.)
  - $x, y \approx \sin^2 \theta_C x$ [SU(3) breaking]$^2$

- For a discussion of local duality [Bigi & Uraltsev], see
  - \text{Nucl. Phys.} B592, 92-106 (2001)
New Physics Mixing Predictions

Possible enhancements to mixing due to new particles and interactions in new physics models
Most new physics predictions for $x$
- Extended Higgs, tree-level FCNC
- Fourth generation down-type quarks
- Supersymmetry: gluinos, squarks
- Lepto-quarks

- Large possible SM contributions to mixing require observation of either a CP-violating signal or $|x| >> |y|$ to establish presence of NP


<table>
<thead>
<tr>
<th>Fourth generation</th>
<th>Vector leptoquarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q = -1/3$ singlet quark</td>
<td>Flavor-conserving Two-Higgs</td>
</tr>
<tr>
<td>$Q = +2/3$ singlet quark</td>
<td>Flavor-changing neutral Higgs</td>
</tr>
<tr>
<td>Little Higgs</td>
<td>Scalar leptoquarks</td>
</tr>
<tr>
<td>Generic $Z'$</td>
<td>MSSM</td>
</tr>
<tr>
<td>Left-right symmetric</td>
<td>Supersymmetric alignment</td>
</tr>
</tbody>
</table>

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Which are the sources of flavour symmetry breaking accessible at low energies?

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SM}} + \sum \frac{c_{ij}}{\Lambda^2} O_{ij}^{(6)} \]

G.I, Nir, Perez '10

<table>
<thead>
<tr>
<th>Operator</th>
<th>Bounds on $\Lambda$ (TeV)</th>
<th>Bounds on $c_{ij}$ ($\Lambda = 1$ TeV)</th>
<th>Observables</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Re</td>
<td>Im</td>
<td>Re</td>
</tr>
<tr>
<td>$(\bar{s}_L \gamma^\mu d_L)^2$</td>
<td>9.8 $\times 10^2$</td>
<td>1.6 $\times 10^4$</td>
<td>9.0 $\times 10^{-7}$</td>
</tr>
<tr>
<td>$(\bar{s}_R d_L)(\bar{s}_L d_R)$</td>
<td>1.8 $\times 10^4$</td>
<td>3.2 $\times 10^5$</td>
<td>6.9 $\times 10^{-9}$</td>
</tr>
<tr>
<td>$(\bar{c}_L \gamma^\mu u_L)^2$</td>
<td>1.2 $\times 10^3$</td>
<td>2.9 $\times 10^3$</td>
<td>5.6 $\times 10^{-7}$</td>
</tr>
<tr>
<td>$(\bar{c}_R u_L)(\bar{c}_L u_R)$</td>
<td>6.2 $\times 10^3$</td>
<td>1.5 $\times 10^4$</td>
<td>5.7 $\times 10^{-8}$</td>
</tr>
<tr>
<td>$(\bar{b}_L \gamma^\mu d_L)^2$</td>
<td>5.1 $\times 10^2$</td>
<td>9.3 $\times 10^2$</td>
<td>3.3 $\times 10^{-6}$</td>
</tr>
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<td>7.6 $\times 10^{-5}$</td>
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New flavor-breaking sources of $O(1)$ at the TeV scale are definitely excluded
Charm Mixing
Time-Evolution of $D^0 \rightarrow K\pi$ Decays

DCS and mixing amplitudes interfere to give a “quadratic” WS decay rate ($x$, $y \ll 1$):

$$\Gamma_{WS}(t) \propto R_D + \sqrt{R_D y'} \left( \frac{t}{\tau} \right) + \left( \frac{x'^2 + y'^2}{4} \right) \left( \frac{t}{\tau} \right)^2$$

where $x' = x \cos \delta + y \sin \delta$ and $y' = y \cos \delta - x \sin \delta$

and $\delta$ is the phase difference between DCS and CF decays.
Rate of WS events clearly increases with time:

\[
\frac{\Gamma_{WS}(t)}{e^{-t/\tau}} \propto R_D + \sqrt{R_D} y' \left( \frac{t}{\tau} \right) + \left( \frac{x'^2 + y'^2}{4} \right) \left( \frac{t}{\tau} \right)^2
\]

(stat. only)
Simplified Fit Strategy & Validation (2007)

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\[
\frac{\Gamma_{\text{WS}}(t)}{e^{-t/\tau}} \propto R_D + \sqrt{R_D} y' \left( \frac{t}{\tau} \right) + \left( \frac{x'^2 + y'^2}{4} \right) \left( \frac{t}{\tau} \right)^2
\]

Inconsistent with no-mixing hypothesis: \( \chi^2 = 24 \)
Simplified Fit Strategy & Validation (2007)

Rate of WS events clearly increases with time:

\[ \frac{\Gamma_{WS}(t)}{e^{-t/\tau}} \propto R_D + \sqrt{R_D} \frac{y'}{\tau} \left( \frac{t}{\tau} \right) + \left( \frac{x'^2 + y'^2}{4} \right) \left( \frac{t}{\tau} \right)^2 \]

Consistent with prediction from full likelihood fit
\[ \chi^2 = 1.5 \]

Inconsistent with no-mixing hypothesis:
\[ \chi^2 = 24 \]
Lifetime Ratio Observables

In the D* tagged analysis, measure:

\[ \tau_{K\pi} \equiv \tau(D^0 \rightarrow K^-\pi^+ + c.c.) \]  
\[ \tau_{D^0} \equiv \tau(D^0 \rightarrow h^-h^+) \]  
\[ \tau_{hh} \]  

CP-mixed right-sign Cabibbo-favored (CF) decay lifetime  
CP-even singly Cabibbo-suppressed (SCS) decay lifetime

Construct mixing variable

\[ y_{CP} \equiv \frac{\tau_{K\pi}}{\tau_{hh}} - 1 \]

where

\[ \tau_{hh} = \frac{\tau_{D^0} + \tau_{D^0}}{2} \]

and CPV asymmetry:

\[ \Delta Y \equiv \frac{\tau_{K\pi}}{\tau_{hh}} A_\tau \]

where

\[ A_\tau = \frac{\tau_{D^0} - \tau_{D^0}}{\tau_{D^0} + \tau_{D^0}} = -A_\tau \]

In the untagged analysis, measure only:

\[ y_{CP} \equiv \frac{\tau_{K\pi}^{RS+WS}}{\tau_{hh}} - 1 \]

where \( \tau_{K\pi}^{RS+WS} \) is the lifetime of the right-sign decay, with a small admixture of wrong sign decays

In the limit of CP conservation, \( y_{CP} = y \) and \( \Delta Y = 0 \)

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These plots illustrate the time-integrated PDF and the average decay time as a function of position in the Dalitz plot for \((x,y) = (0.16\%, 0.57\%)\). The sizes of the boxes in the right-hand plot reflect the number of entries, and the colors reflect the average decay time.
Fit to all time-dependent CPV measurements.

CPV-allowed plot, no mixing \((x, y) = (0, 0)\) point: \(\Delta \chi^2 = 109.6\), \(CL = 1.56 \times 10^{-24}\), no mixing excluded at 10.2σ

No CPV \((|q/p|, \varphi) = (1, 0)\) point:
\(\Delta \chi^2 = 1.218\), \(CL = 0.456\), consistent with CP conservation
SuperB Accelerator Design Goals & Issues

\[ \mathcal{L} = 10^{36} \text{ cm}^{-2} \text{ sec}^{-1}; \quad \int \mathcal{L} \, dt = 75 \text{ ab}^{-1} \quad \text{at the } \Upsilon(4S) \]
\[ \mathcal{L} = 10^{35} \text{ cm}^{-2} \text{ sec}^{-1}; \quad \int \mathcal{L} \, dt = 500 \text{ fb}^{-1} \quad \text{at the } \Psi(3770) \]

Baseline Design

• 6.7 GeV $e^+ \times 4.18$ GeV $e^-$ ($\beta\gamma \sim 0.24$)
• 1892 mA $\times$ 2410 mA
• 80% polarization of the electron beam
• beam size is $\sim 7$ µm horizontal $\times$ 35 nm vertical
• total RF power is 17 MW
• luminosity lifetimes are 4.82 & 6.14 minutes
  beam lifetimes $\sim 4$ minutes
• circumference 1258 meters (fits onto LNF site)
• designed to re-use PEP-II magnets and RF

Full funding approved by Italian Parliament
Correlated $D$ decays at the $\Psi(3770)$

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To calculate correlated $D$ decay rates at the $J/\Psi(3770)$ we calculate the correlated amplitude for the $D$ and the $\bar{D}$ to decay to the states $\alpha$ and $\beta$ at times $t_1$ and $t_2$ respectively, where the times are measured in the center-of-mass (CM) system and $t = 0$ is the time of the $e^+e^- \rightarrow c\bar{c}$ production. Because the $\Psi(3770)$ is $J^{PC} = 1^{--}$ state, we antisymmetrize the amplitude with respect to charge conjugation.

$$M = \frac{1}{\sqrt{2}} \left[ \langle \alpha | \mathcal{H} | D^0(t_1) \rangle \langle \beta | \mathcal{H} | \bar{D}^0(t_2) \rangle - \langle \beta | \mathcal{H} | D^0(t_2) \rangle \langle \alpha | \mathcal{H} | \bar{D}^0(t_1) \rangle \right]$$

(1)

The time evolution of the $D^0-\bar{D}^0$ system is described by the Schrödinger equation

$$i \frac{\partial}{\partial t} \begin{pmatrix} D^0(t) \\ \bar{D}^0(t) \end{pmatrix} = \begin{pmatrix} M - i \Gamma \\ 2 \end{pmatrix} \begin{pmatrix} D^0(t) \\ \bar{D}^0(t) \end{pmatrix},$$

(2)

where the $M$ and $\Gamma$ matrices are Hermitian, and $CPT$ invariance requires $M_{11} = M_{22} \equiv M$ and $\Gamma_{11} = \Gamma_{22} \equiv \Gamma$. 
**Some Notation**

The two eigenstates $D_1$ and $D_2$ of the effective Hamiltonian are

$$|D_{1,2}\rangle = p|D^0\rangle \pm q|\bar{D}^0\rangle , \quad |p|^2 + |q|^2 = 1 .$$

(3)

The corresponding eigenvalues are

$$\lambda_{1,2} \equiv m_{1,2} - \frac{i}{2} \Gamma_{1,2} = \left( M - \frac{i}{2} \Gamma \right) \pm \frac{q}{p} \left( M_{12} - \frac{i}{2} \Gamma_{12} \right) ,$$

(4)

where $m_{1,2}, \Gamma_{1,2}$ are the masses and decay widths and

$$\frac{q}{p} = \sqrt{\frac{M_{12}^{\ast} - \frac{i}{2} \Gamma_{12}^{\ast}}{M_{12}^{\ast} - \frac{i}{2} \Gamma_{12}}} .$$

(5)

The proper time evolution of the eigenstates of Eq. 2 is

$$|D_{1,2}(t)\rangle = e_{1,2}(t)|D_{1,2}\rangle , \quad e_{1,2}(t) = e^{-i(m_{1,2} - \frac{i}{2} \Gamma_{1,2})t} .$$

(6)

A state that is prepared as a flavor eigenstate $|D^0\rangle$ or $|\bar{D}^0\rangle$ at $t = 0$ will evolve according to

$$|D^0(t)\rangle = \frac{1}{2p} \left[ p(e_1(t) + e_2(t))|D^0\rangle + q(e_1(t) - e_2(t))|\bar{D}^0\rangle \right]$$

(7)

$$|\bar{D}^0(t)\rangle = \frac{1}{2q} \left[ p(e_1(t) - e_2(t))|D^0\rangle + q(e_1(t) + e_2(t))|\bar{D}^0\rangle \right] .$$

(8)

We adopt a version of the standard notation

$$\Gamma = \frac{\Gamma_1 + \Gamma_2}{2} , \quad x = \frac{m_1 - m_2}{\Gamma} , \quad y = \frac{\Gamma_1 - \Gamma_2}{2\Gamma} .$$

(9)
Forms of $\mathcal{M}$ and $|\mathcal{M}|^2$

After a bit of algebra we can write the matrix element as

$$2\sqrt{2} \mathcal{M} = \left( \frac{q}{p} \mathcal{A}_\alpha \mathcal{A}_\beta - \frac{p}{q} \mathcal{A}_\alpha \mathcal{A}_\beta \right) [e_1(t_1)e_2(t_2) - e_1(t_2)e_2(t_1)]$$

\[ + (\mathcal{A}_\alpha \overline{\mathcal{A}}_\beta - \overline{\mathcal{A}}_\alpha \mathcal{A}_\beta) [e_1(t_1)e_2(t_2) + e_1(t_2)e_2(t_1)] \]

which has the form

$$2\sqrt{2} \mathcal{M} = X(e_{11}e_{22} - e_{12}e_{21}) + Y(e_{11}e_{22} + e_{12}e_{21}).$$

From this one calculates

$$8|\mathcal{M}|^2 = e^{-\Gamma(t_1+t_2)} \times \left\{ XX^* (\cosh y\Gamma\Delta t - \cos x\Gamma\Delta t) \right. - 2 \Re(XY^*) \sinh y\Gamma\Delta t + 2 \Im(XY^*) \sin x\Gamma\Delta t$$
\[ + YY^* (\cosh y\Gamma\Delta t + \cos x\Gamma\Delta t) \right\} \]

For $x\Gamma\Delta t, y\Gamma\Delta t \ll 1$ this can be approximated by

$$4|\mathcal{M}|^2 = e^{-\Gamma(t_1+t_2)} \times \left\{ XX^* \left[ \frac{(x^2 + y^2)}{4} \right](\Gamma\Delta t)^2 \right. - \Re(XY^*) y\Gamma\Delta t + \Im(XY^*) x\Gamma\Delta t$$
\[ + YY^* \left[ 1 + \frac{(y^2 - x^2)}{4} \right](\Gamma\Delta t)^2 \right\} \]

- $Y$ is the unmixed amplitude
- $X$ is the mixing amplitude
- $XY^*$ controls the interference terms in the mixing rate

see also Zhi-zhong Xing, Phys.Rev. D55 (1997) 196-218
Correlated \((K^-K^+, \, K^-K^+)\) decays

\textit{CP even versus CP even}

For two \textit{CP}-even eigenstates \(\alpha\) and \(\beta\),

\[
Y = 0
\]

\[
X = \left(\frac{q}{p} - \frac{p}{q}\right) A_\alpha A_\beta.
\]

so the rate is

\[
|\mathcal{M}|^2 = e^{-\Gamma(t_1+t_2)} \times \left|\frac{q}{p} - \frac{p}{q}\right|^2 \left|A_\alpha\right|^2 \left|A_\beta\right|^2 \left(\frac{x^2 + y^2}{4}\right) (\Gamma \Delta t)^2.
\]

In the limit that \textit{CP} is a good symmetry, this rate goes to zero. To estimate what might be possible at Super \(B\), we take the numbers of \(K^\mp \pi^\pm\) versus \textit{CP} even events observed by CLEO-c (605), scale by the approximate ratio of \(K^-K^+\) plus \(\pi^--\pi^+\) events observed \((\approx 0.13)\) [to account for the value of \(|A_\alpha|^2 |A_\beta|^2\)], and scale by the nominal relative luminosity. This procedure gives approximately \(120K\) as the coefficient of \((x^2 + y^2) (\Gamma \Delta t)^2/4\). Using \((x^2 + y^2) (\Gamma \Delta t)^2/2\) as an estimate of the time integral, and taking \(x^2 + y^2 = 10^{-4}\), the integrated signal will be about

\[
\left|\frac{q}{p} - \frac{p}{q}\right|^2 \times 6 \text{ events}.
\]
Correlated \((K^-\pi^+\), \(K^-\pi^-\)) decays

A similar result obtains for common final states such as \(K^-\pi^+\). If \(\alpha = \beta\) then \(A_\beta = A_\alpha\) and \(A_\beta = \overline{A}_\alpha\). Again, the unmixed amplitude goes to zero. However, the pure mixing term does not require \(CP\) violation to be non-zero.

\[
Y = 0 \tag{17}
\]

\[
X = \left( \frac{q}{p} A_\alpha \overline{A}_\alpha - \frac{p}{q} A_\alpha A_\alpha \right). \tag{18}
\]

In this case, \(A_\alpha\) corresponds to the Cabibbo-favored amplitude and \(\overline{A}_\alpha\) to the doubly Cabibbo-suppressed amplitude. With \(\overline{A}_\alpha = ke^{i\delta}A_\alpha\) the rate can be written

\[
|M|^2 = e^{-\Gamma(t_1+t_2)} \times \left| \frac{q}{p} k^2 e^{i2\delta} - \frac{p}{q} \right|^2 |A_\alpha|^2 |A_\alpha|^2 \left( \frac{x^2 + y^2}{4} \right) (\Gamma \Delta t)^2. \tag{18}
\]

As a first approximation, we can ignore both the doubly Cabibbo-suppressed amplitude and \(CP\) violation. In this case

\[
|M|^2 \approx e^{-\Gamma(t_1+t_2)} \times |A_\alpha|^2 |A_\alpha|^2 \left( \frac{x^2 + y^2}{4} \right) (\Gamma \Delta t)^2. \tag{19}
\]

CLEO-c observes 600 \(K^-\pi^+\), \(K^+\pi^-\) events, which corresponds to \(2 |A_\alpha|^2 |A_\alpha|^2\). Scaling by relative luminosities, and again using \(10^{-4}\) for \((x^2 + y^2)\), we can project a mixing signal of 23 events in this channel and a similar number in \(K^+\pi^-\) versus \(K^+\pi^-\). While differences nominally can be due to direct \(CP\) violation, indirect \(CP\) violation, or statistical fluctuation, given the existing HFAG bounds on direct and indirect \(CP\) violation, any variation we observe in this channel will be predominantly due to statistical fluctuations.
Correlated \((K\ell\nu, K\ell\nu)\) decays

For opposite-sign semileptonic decays we can choose \(\alpha = K^-\ell^+\nu\) and \(\beta = K^+\ell^-\nu\) for which

\[
Y = A_\alpha \bar{A}_\beta; \quad X = 0
\]  

(20)

The rate is proportional to

\[
|\mathcal{M}|^2 = e^{-\Gamma(t_1+t_2)} \times |A_\alpha|^2 |A_\beta|^2 \left[1 + \frac{(y^2 - x^2)}{4}(\Gamma \Delta t)^2\right].
\]  

(21)

The only signature of mixing in this final state is the quadratic departure from purely exponential decay which is proportional to \((y^2 - x^2)\). This is less than one part in \(10^4\), significantly less than the rate of statistical fluctuations. This final state has no sensitivity to \(CP\) violation in mixing \((q/p \neq 1)\).

For same-sign semileptonic decays we can choose \(\alpha = \beta = K^-\ell^+\nu\). In this case

\[
Y = 0 \quad X = -\frac{p}{q} \left(\mathcal{A}(D^0 \to K^-e^+\nu_e)\right).
\]  

(22)

The corresponding rate is

\[
|\mathcal{M}|^2 = e^{-i\Gamma(t_1+t_2)} \left|\frac{p}{q}\right|^2 A_\alpha \bar{A}_\beta \left|\frac{x^2 + y^2}{4}\right| (\Gamma \Delta t)^2.
\]  

(23)

Extrapolating from CLEO-c’s opposite-sign \(K\pi\) rate, we estimate 23 mixing events in each of \(K^-e^+\nu_e\) versus \(K^-e^+\nu_e\) and \(K^+e^-\nu_e\) versus \(K^+e^-\nu_e\).
Correlated \((K\ell\nu, K^-K^+)\) decays

The correlated decays of \(D^0\bar{D}^0\) into a \(CP\) eigenstate and and semileptonic final state are also (relatively) easy to understand. Consider \(A_\alpha = A(D^0 \rightarrow K^- e^+\nu_e)\) and \(A_\beta = A(D^0 \rightarrow K^-K^+)\) as an example such a final state. In this case

\[ Y = A_\alpha A_\beta; \quad X = -\frac{p}{q} A_\alpha A_\beta \quad (24) \]

The (small \(y\Gamma\Delta t\), small \(x\Gamma\Delta t\)) limit for \((K^-\ell^+X, K^-K^+)\) is

\[ |\mathcal{M}|^2 = e^{-\Gamma(t_1+t_2)} |A_\alpha|^2 |A_\beta|^2 \times \left\{ 1 \pm \Re\left(\frac{p}{q}\right) y\Gamma\Delta t \pm \Im\left(\frac{p}{q}\right) x\Gamma\Delta t + \frac{y^2}{2}(\Gamma\Delta t)^2 \right\}. \quad (25) \]

For \(D^0 \rightarrow K^+\ell^-X\) detected in conjunction with a \(CP\) even final state, \((-p/q)\) in \(XY^*\) becomes \((+q/p)\) and \(A_\alpha = A(\bar{D}^0 \rightarrow K^+\ell^-X)\). As a first approximation, the difference between positive and negative decay time distributions will be proportional to

\[ \left( \Re\left(\frac{p}{q}\right) y - \Im\left(\frac{p}{q}\right) x \right) \times \Gamma |\Delta t| = y'\Gamma |\Delta t| \quad (26) \]

for \(D^0 \rightarrow K^-\ell^+X\) and to

\[ \left( \Re\left(\frac{q}{p}\right) y - \Im\left(\frac{q}{p}\right) x \right) \times \Gamma |\Delta t| = y''\Gamma |\Delta t| \quad (27) \]

for \(\bar{D}^0 \rightarrow K^+\ell^-X\). For \(q/p \approx 1\), the difference between \(y'\) and \(y''\) measures \(|q/p|\).
Correlated \((K^-\ell^+\nu, K^-\pi^+)\) decays

The correlated decays to a semileptonic final state and a hadronic non-\(CP\) eigenstate are somewhat more complicated. For the final state \((K^-\pi^+, K^-e^+\nu_e)\) we can write

\[
A_\alpha = A(D^0 \to K^-\pi^+) \quad \overline{A}_\alpha = ke^{i\delta_{K\pi}} A_\alpha \\
A_\beta = A(D^0 \to K^-e^+\nu_e) \quad \overline{A}_\beta = 0
\]

where \(a, \delta, \phi, k\) and \(\delta_{K\pi}\) are real numbers. Writing The factor \(k \approx \tan^2 \theta_C\) is the ratio of the magnitudes of the doubly Cabibbo-suppressed (DCS) and Cabibbo-favored (CF) amplitudes. The angle \(\delta_{K\pi}\) is the relative strong phase between the CF and DCS amplitudes to the same final state. The mixing and direct amplitudes for \((K^-\pi^+, K^-e^+\nu_e)\) are

\[
X = -\frac{p}{q} A_{\alpha} A_{\beta} \quad Y = ke^{i\delta_{K\pi}} A_{\alpha} A_{\beta}
\]

The (small \(y\Gamma\Delta t\), small \(x\Gamma\Delta t\)) limit for the \((K^-\ell^+X, K^-\pi^+)\) decay rate is

\[
|\mathcal{M}|^2 = \frac{1}{4} e^{-\Gamma(t_1+t_2)} |A_\alpha|^2 |A_\beta|^2 \times \left\{ \frac{|p|^2}{q} \left( \frac{x^2 + y^2}{4} \right) (\Gamma\Delta t)^2 \right. \\
- \left( \Re\left(\frac{p}{q}\right) \cos \delta_{K\pi} + \Im\left(\frac{p}{q}\right) \sin \delta_{K\pi} \right) k y \Gamma\Delta t \\
+ \left( \Im\left(\frac{p}{q}\right) \cos \delta_{K\pi} - \Re\left(\frac{p}{q}\right) \sin \delta_{K\pi} \right) k x \Gamma\Delta t \\
+ k^2 \left[ 1 + \left( \frac{y^2 - x^2}{4} \right) (\Gamma\Delta t)^2 \right] \right\}. \tag{28}
\]
Correlated \((K^-\pi^+, K^-K^+)\) decays

The correlated decays to a \(CP\) eigenstate and a hadronic non-\(CP\) eigenstate are somewhat more complicated. Consider, as a first example, the final state \((K^-\pi^+, K^-K^+)\). We can write

\[
\mathcal{A}_\alpha = \mathcal{A}(D^0 \to K^-\pi^+) \quad \mathcal{A}_\beta = \mathcal{A}(D^0 \to K^-K^+) \quad \bar{\mathcal{A}}_\alpha = k e^{i\delta_{K\pi}} \mathcal{A}_\alpha \quad \bar{\mathcal{A}}_\beta = \mathcal{A}_\beta
\]

The mixing and direct amplitudes for \((K^-\pi^+, K^-K^+)\) are

\[
X = \left( \frac{q}{p} \right) k e^{i\delta_{K\pi}} \mathcal{A}_\alpha \mathcal{A}_\beta \quad Y = (1 - k e^{i\delta_{K\pi}}) \mathcal{A}_\alpha \bar{\mathcal{A}}_{\beta}
\]

As is well-known, the time-integrated rate is dominated by the term

\[
YY^* = (1 - 2k \cos \delta_{K\pi} + k^2) \mathcal{A}_\alpha \bar{\mathcal{A}}_\alpha \mathcal{A}_\beta \bar{\mathcal{A}}_\beta
\]

which depends linearly on \(\cos \delta_{K\pi}\).

The real and imaginary parts of the interference term are

\[
\Re(XY^*) = k \left( 1 + \left| \frac{q}{p} \right|^2 \right) \left[ \Re \left( \frac{p}{q} \right) \cos \delta - \Im \left( \frac{p}{q} \right) \sin \delta \right] - \Re \left( \frac{p}{q} \right) (1 + k^2)
\]

\[
\Im(XY^*) = k \left( 1 - \left| \frac{q}{p} \right|^2 \right) \left[ \Im \left( \frac{p}{q} \right) \cos \delta + \Re \left( \frac{p}{q} \right) \sin \delta \right] - \Im \left( \frac{p}{q} \right) (1 + k^2)
\]
Correlated \((K^-\ell^+X, K^-\pi^+\pi^0)\) decays

With the notation
\[
\mathcal{A}(D^0 \to K^-\pi^+\pi^0) = A_r \zeta(s_{12}, s_{13}) \tag{31}
\]
\[
\bar{\mathcal{A}}(D^0 \to K^-\pi^+\pi^0) = \bar{A}_r \bar{\zeta}(s_{12}, s_{13}) = ke^{i\delta_{K\pi\pi^0}}A_r \bar{\zeta}(s_{12}, s_{13})
\]

The (small \(y\Gamma\Delta t\), small \(x\Gamma\Delta t\)) limit for the \((K^-\ell^+X, K^-\pi^+\pi^0)\) decay rate is

\[
|M|^2 = \frac{1}{4} e^{-\Gamma(t_1+t_2)} |A_r|^2 |A_\beta|^2 \times \tag{32}
\]
\[
\left\{ \left| \frac{p}{q} \right|^2 \zeta(s_{12}, s_{13}) \zeta^*(s_{12}, s_{13}) \left( \frac{x^2 + y^2}{4} \right) (\Gamma\Delta t)^2 \right. \\
- \left[ \mathcal{R} \left( \frac{p}{q} \zeta(s_{12}, s_{13}) \zeta^*(s_{12}, s_{13}) \right) \cos \delta_{K\pi\pi^0} \right] k y \Gamma\Delta t \\
+ \left[ \mathcal{I} \left( \frac{p}{q} \zeta(s_{12}, s_{13}) \zeta^*(s_{12}, s_{13}) \right) \sin \delta_{K\pi\pi^0} \right] k y \Gamma\Delta t \\
+ \left[ \mathcal{R} \left( \frac{p}{q} \zeta(s_{12}, s_{13}) \zeta^*(s_{12}, s_{13}) \right) \cos \delta_{K\pi\pi^0} \right] k x \Gamma\Delta t \\
- \left[ \mathcal{I} \left( \frac{p}{q} \zeta(s_{12}, s_{13}) \zeta^*(s_{12}, s_{13}) \right) \sin \delta_{K\pi\pi^0} \right] k x \Gamma\Delta t \\
\left. + k^2 \zeta(s_{12}, s_{13}) \zeta^*(s_{12}, s_{13}) \left[ 1 + \left( \frac{y^2 - x^2}{4} \right) (\Gamma\Delta t)^2 \right] \right\}.
\]

As a first approximation, the time-integrated rate is dominated by the doubly-Cabibbo suppressed rate associated with \(YY^*\). To a lesser degree, the pure mixing rate proportional to the Cabibbo favored rate, \(XX^*\), also contributes.
Correlated \((K^-K^+, K^-\pi^+\pi^0)\) decays

In this case, the time-integrated rate will be dominated by

\[
YY^* = \zeta\zeta^* + k^2\bar{\zeta}\bar{\zeta}^* - 2k \left[ \Re(\bar{\zeta}\bar{\zeta}^*) \cos \delta_{K\pi\pi^0} + \Im(\bar{\zeta}\bar{\zeta}^*) \sin \delta_{K\pi\pi^0} \right]. \tag{33}
\]

The time-odd rate will depend on the real and imaginary parts of

\[
XY^* = -\frac{q}{p}k^2\zeta\bar{\zeta}^* - \frac{p}{q}\zeta\zeta^*
\]

\[
+ k \left[ \frac{q}{p}e^{i\delta_{K\pi\pi^0}}\bar{\zeta}\zeta^* + \frac{p}{q}e^{-i\delta_{K\pi\pi^0}}\bar{\zeta}\bar{\zeta}^* \right]. \tag{34}
\]

In the limit \(p/q = 1\),

\[
XY^* \to -k^2\zeta\bar{\zeta}^* - \zeta\zeta^* + 2k \left[ \Re(\bar{\zeta}\bar{\zeta}^*) \cos \delta_{K\pi\pi^0} + \Im(\bar{\zeta}\bar{\zeta}^*) \sin \delta_{K\pi\pi^0} \right] \tag{35}
\]

which is purely real and equal in magnitude to \(YY^*\). In this limit, the time-odd part of the rate is proportional only to \(y\Gamma\Delta t\) and is independent of \(x\).
Correlated \((K^-\pi^+, K^-\pi^+\pi^0)\) decays

Here we will write

\[
A_\alpha = A(D^0 \rightarrow K^-\pi^+) \\
\overline{A}_\alpha = k_1 e^{i\delta_1} A_\alpha \\
A_\beta = A(D^0 \rightarrow K^-\pi^+\pi^0) = A_r \zeta \\
\overline{A}_\beta = k_2 e^{i\delta_2} A_r \overline{\zeta}
\]

so that

\[
X = \left(\frac{q}{p} k_1 k_2 e^{i(\delta_1+\delta_2)} \overline{\zeta} - \frac{p}{q} \zeta\right) A_r A_\beta \\
Y = (k_2 e^{i\delta_2} \overline{\zeta} - k_1 e^{i\delta_1} \zeta) A_r A_\beta.
\]

It follows that

\[
YY^* = k_2^2 \overline{\zeta} \zeta^* + k_1^2 \zeta \overline{\zeta}^* \\
-2 k_1 k_2 \left[\Re(\zeta \overline{\zeta}^*) \cos(\delta_1 - \delta_2) - \Im(\zeta \overline{\zeta}^*) \sin(\delta_1 - \delta_2)\right] |A_\alpha|^2 |A_r|^2,
\]

and as a good first approximation,

\[
XY^* \approx -\frac{p}{q} \left\{k_2 \cos \delta_2 \Re(\zeta \overline{\zeta}^*) + k_2 \sin \delta_2 \Im(\zeta \overline{\zeta}^*) - k_1 \cos \delta_1 \zeta \overline{\zeta}^*\right\} |A_\alpha|^2 |A_r|^2.
\]
Some Notation for $D^0 \to K_S^0 \pi^- \pi^+$

At first sight, $K_S^0 \pi^- \pi^+$, appears to be similar to $K^- \pi^+ \pi^0$ as both are three-body decays whose amplitudes are often described using isobar models. However, in the limit of no direct $CP$ violation in $D$ decay, and ignoring the known $CP$ violation in $K_S^0$ decay, we can exploit the relationship

$$A(D^0 \to K_S^0 \pi^- \pi^+)(s_{12}, s_{13}) = A(D^0 \to K_S^0 \pi^- \pi^+)(s_{13}, s_{12})$$  \hspace{1cm} (40)

Using the notation

$$A(D^0 \to K_S^0 \pi^- \pi^+)(s_{13}, s_{12}) = A_r \zeta(s_{12}, s_{13})$$  \hspace{1cm} (41)

and assuming no direct $CP$ violation, we have

$$A_\alpha = A_r \zeta(s_{12}, s_{13}); \quad \bar{A}_\alpha = A_r \zeta(s_{13}, s_{12})$$  \hspace{1cm} (42)

It is sometimes useful to re-write $\zeta(s_{12}, s_{13})$ and $\zeta(s_{13}, s_{12})$ in terms of symmetric and antisymmetric functions

$$\zeta_S(s_{13}, s_{12}) = \frac{1}{2} [\zeta(s_{12}, s_{13}) + \zeta(s_{13}, s_{12})]$$  \hspace{1cm} (43)

$$\zeta_A(s_{13}, s_{12}) = \frac{1}{2} [\zeta(s_{12}, s_{13}) - \zeta(s_{13}, s_{12})]$$

so that

$$\zeta(s_{12}, s_{13}) = \zeta_S(s_{13}, s_{12}) + \zeta_A(s_{13}, s_{12})$$  \hspace{1cm} (44)

$$\zeta(s_{13}, s_{12}) = \zeta_S(s_{13}, s_{12}) - \zeta_A(s_{13}, s_{12}).$$

Note that we can use the same notation for $D^0 \to \pi^0 \pi^- \pi^+$. 


Correlated \((K^-\ell^+\nu, K_S^0\pi^-\pi^+)\) decays

With the notation introduced for \(A_\alpha = A(D^0 \to K_S^0\pi^-\pi^+)\),

\[
X = -\frac{p}{q}(\zeta_S + \zeta_A)A_rA_\beta \\
Y = -(\zeta_S - \zeta_A)A_rA_\beta
\]

which gives

\[
YY^* = (2\zeta_S\zeta_S^* + 2\zeta_A\zeta_A^* - \zeta\zeta^*) |A_r|^2 |A_\beta|^2 \\
XY^* = \frac{p}{q} [\zeta_S\zeta_S^* - \zeta_A\zeta_A^* - 2i\Im(\zeta_S\zeta_A^*)] |A_r|^2 |A_\beta|^2
\]

so that

\[
\Re(XY^*) = \left[ \Re\left(\frac{p}{q}\right)(\zeta_S\zeta_S^* - \zeta_A\zeta_A^*) + 2\Im\left(\frac{p}{q}\right)\Im(\zeta_S\zeta_A^*) \right] |A_r|^2 |A_\beta|^2 \\
\Im(XY^*) = \left[ -2\Re\left(\frac{p}{q}\right)\Im(\zeta_S\zeta_A^*) + \Im\left(\frac{p}{q}\right)(\zeta_S\zeta_S^* - \zeta_A\zeta_A^*) \right] |A_r|^2 |A_\beta|^2
\]
Correlated \((K^- K^+, K_S^0 \pi^- \pi^+)\) decays

As usual, the time integrated rate is dominated by

\[ YY^* = 4 \zeta_A(s_{12}, s_{13}) \zeta_A^*(s_{12}, s_{13}) |A_r|^2 |A|^2 \] (48)

which we can identify as the antisymmetric rate. Were we to consider \(K_S^0 \pi^- \pi^+\) produced in conjunction with a pure CP odd eigenstate rather than CP even, \(YY^*\) would be the symmetric rate instead. The time-odd rates are proportional to the real and imaginary parts of \(XY^*\) which is

\[ XY^* = 2 \left[ \zeta_S \zeta_A^* \left( \frac{p}{q} - \frac{q}{p} \right) + \zeta_A \zeta_A^* \left( \frac{p}{q} + \frac{q}{p} \right) \right] |A_r|^2 |A|^2. \] (49)

In the limit \(p = q\), \(XY^* \rightarrow YY^*\). Were we to consider \(K_S^0 \pi^- \pi^+\) produced in conjunction with a pure CP odd eigenstate rather than CP even, \(XY^*\) becomes

\[ XY^* = 2 \left[ \left( \zeta_S \zeta_A^* \right)^* \left( \frac{p}{q} - \frac{q}{p} \right) + \zeta_S \zeta_S^* \left( \frac{p}{q} + \frac{q}{p} \right) \right] |A_r|^2 |A|^2. \] (50)

The roles of \(\zeta_S\) and \(\zeta_A\) are interchanged. If \(p \neq q\) the \(\Re(\zeta_S \zeta_A^*) = \Re(\zeta_S \zeta_A^*)^*\) but the \(\Im(\zeta_S \zeta_A^*) = -\Im(\zeta_S \zeta_A^*)^*\) so the time-odd asymmetries will differ and the difference of the two as a function of position in the Dalitz plot will provide additional sensitivity to the real and imaginary parts of \(p/q\).
Correlated \((K^−\pi^+, K_{S}^{0}\pi^−\pi^+)\) decays

With the same type of notation as used earlier,
\[
YY^* = \zeta \zeta'^* + k^2 \zeta \zeta^* - 2k \Re (e^{i\delta} \zeta \zeta'^*)
\]
\[
= \zeta' \zeta'^* + k^2 \zeta \zeta^* - 2k \cos \delta \Re (\zeta \zeta'^*) + 2k \sin \delta \Im (\zeta \zeta'^*).
\] (51)

We can identify the real and imaginary parts of \(\zeta \zeta'^*\) with \(\zeta_S\) and \(\zeta_A\) writing
\[
\zeta \zeta'^* = \zeta_S \zeta^* - \zeta_A \zeta'^* - 2i \Im (\zeta_S \zeta_A^*)
\] (52)

from which we find
\[
\Re (\zeta \zeta'^*) = \zeta_S \zeta^* - \zeta_A \zeta'^*; \quad \Im (\zeta \zeta'^*) = -2 \Im (\zeta_S \zeta_A^*).
\] (53)

This gives

\[
YY^* = \zeta' \zeta'^* + k^2 \zeta \zeta^* - 2k \cos \delta (\zeta_S \zeta^* - \zeta_A \zeta'^*) - 4k \sin \delta \Im (\zeta_S \zeta_A^*).
\] (54)

The time-odd rate is proportional to the real and imaginary parts of
\[
XY^* = \frac{q}{p} k^2 (\zeta \zeta'^*) + \frac{p}{q} (\zeta \zeta'^*)
\]
\[
+ k \left[ \frac{q}{p} e^{i\delta} (\zeta' \zeta'^*) - \frac{p}{q} e^{-i\delta} (\zeta \zeta^*) \right]
\] (55)

In the limit \(p/q \to 1\), these become
\[
\Re (XY^*) = (1 + k^2) [\zeta_S \zeta^* - \zeta_A \zeta_A^*] + \cos \delta [\zeta \zeta^* - k \zeta' \zeta'^*]
\]
\[
\Im (XY^*) = \sin \delta [\zeta \zeta^* + k \zeta' \zeta'^*] - (1 - k^2) \Im (\zeta_S \zeta_A^*)
\] (56)
Correlated \( (K^-\pi^+\pi^0, K^-\pi^+\pi^0) \) decays

For this correlated final state we will use the notation
\[
A_\alpha(s_{12}, s_{13}) = A_r \zeta(s_{12}, s_{13})
\]

\[
\bar{A}_\alpha(s_{12}, s_{13}) = A_r \bar{\zeta}(s_{12}, s_{13}) = \kappa(s_{12}, s_{13}) e^{i\epsilon(s_{12}, s_{13})} A_r \zeta(s_{12}, s_{13})
\]

\[
A_\beta(s'_{12}, s'_{13}) = A_r \zeta'(s'_{12}, s'_{13})
\]

\[
\bar{A}_\beta(s'_{12}, s'_{13}) = A_r \bar{\zeta}'(s'_{12}, s'_{13}) = \kappa'(s'_{12}, s'_{13}) e^{i\epsilon(s'_{12}, s'_{13})} A_r \zeta'(s'_{12}, s'_{13}).
\]

The real functions \( \kappa, \kappa', \epsilon, \) and \( \epsilon' \) are chosen so that \( \kappa \) and \( \kappa' \) are positive definite.

\[
YY^* = \zeta \zeta^* \bar{\zeta} \bar{\zeta}'^* + \bar{\zeta} \zeta^* \zeta' \zeta'^* + 2\Re \left[ (\zeta \bar{\zeta}^*) (\zeta' \bar{\zeta}'^*) \right] |A_r|^4.
\]

The first two terms are the products of the Cabibbo favored rate for one decay and the doubly-Cabibbo suppressed rate for the other. The last term is the product of two Cabibbo favored, doubly-Cabibbo suppressed interference rates. As a good approximation, we can calculate the interference term ignoring the doubly-Cabibbo suppressed term in \( X \):

\[
XY^* \approx \frac{p}{q} \left[ (\zeta \zeta^*) (\zeta' \bar{\zeta}'^*) - (\zeta \bar{\zeta}^*) (\zeta' \zeta'^*) \right] |A_r|^4.
\]

Here, each term is the product of a Cabibbo favored rate for one decay and the interference of amplitudes for the other. Events will populate a four-dimensional phase space corresponding to the two Dalitz plot positions \( (s_{12}, s_{13}) \) and \( (s'_{12}, s'_{13}) \). Furthermore, this interference term is antisymmetric under the interchange of the \( \zeta \) and \( \zeta' \). This is evident algebraically from the form of Eqn. (59). Physically, it corresponds to identifying one or the other Dalitz plot position as that of the first \( D \) to decay. As the interference rate is time-odd, \( XY^* \) must be antisymmetric when the two Dalitz plot positions are interchanged.
Correlated $K^0_S \pi^- \pi^+$, $K^0_S \pi^- \pi^+$ decays

Here, we use notation here more similar to that used for $K^- \pi^+ \pi^0$, $K^- \pi^+ \pi^0$ than for $K^0_S \pi^- \pi^+$, $K^- \pi^+$:

$$
\mathcal{A}_\alpha(s_{12}, s_{13}) = \zeta(s_{12}, s_{13}) A_r = (\zeta_S + \zeta_A) A_r \quad (60)
$$

$$
\overline{\mathcal{A}}_\alpha(s_{12}, s_{13}) = \overline{\zeta}(s_{13}, s_{12}) A_r = (\zeta_S - \zeta_A) A_r
$$

$$
\mathcal{A}_\beta(s'_{12}, s'_{13}) = \zeta(s'_{12}, s'_{13}) A_r = (\zeta'_S + \zeta'_A) A_r
$$

$$
\overline{\mathcal{A}}_\beta(s'_{12}, s'_{13}) = \overline{\zeta}(s'_{13}, s'_{12}) A_r = (\zeta'_S - \zeta'_A) A_r
$$

The prime superscript ($'$) distinguishes the amplitudes associated with the two Dalitz plot positions of the $K^0_S \pi^- \pi^+$ decays rather than the amplitudes associated with direct $D^0$ and $D^0$ decay. With this notation

$$
Y Y^* = 4 \left\{ \zeta_A \zeta'_A \zeta'_S \zeta^*_S + \zeta_S \zeta'^*_S \zeta'_A \zeta^*_A \\
-2 \left[ \Re(\zeta_S \zeta^*_A) \Re(\zeta'_S \zeta'^*_A) + \Im(\zeta_S \zeta^*_A) \Im(\zeta'_S \zeta'^*_A) \right] \right\} \quad (61)
$$

In the limit $p = q$, the mixing amplitude becomes

$$
X = (\zeta_S - \zeta_A) (\zeta'_S - \zeta'_A) - (\zeta_S + \zeta_A) (\zeta'_S + \zeta'_A) \quad (62)
$$

$$
= -2 [\zeta_A \zeta'_S + \zeta_S \zeta'_A]
$$

in which case

$$
X Y^* = -4 (\zeta_A \zeta'_S + \zeta_S \zeta'_A) (\zeta^*_A \zeta'^*_S - \zeta S \zeta'^*_A) \quad (63)
$$

$$
= -4 \left[ \zeta_A \zeta'^*_S \zeta'_S \zeta'_A - \zeta_S \zeta^*_A \zeta'_A \zeta'^*_S + 2i \Im(\zeta_S \zeta^*_A \zeta'_A \zeta'^*_S) \right]
$$

$$
= -4 \left[ \zeta_A \zeta'^*_S \zeta'_S \zeta'_A - \zeta_S \zeta^*_A \zeta'_A \zeta'^*_S + 2i \left( -\Re(\zeta_S \zeta^*_A) \Im(\zeta_S \zeta'^*_A) + \Re(\zeta'_S \zeta'^*_A) \Im(\zeta_S \zeta_A) \right) \right]
$$
Sensitivities - As Good As It Gets

Use a Toy Monte Carlo procedure to generate and fit datasets

- extrapolate (roughly) from numbers of events seen by CLEO-c to 500 fb$^{-1}$
- assume we know all amplitudes and related terms exactly: $XX^*$, $YY^*$, $\Re(XY^*)$, $\Im(XY^*)$
- assume the quadratic expansion for time-dependence
- calculate expected numbers of events in each (Dalitz plot bin) × (time bin). Use 100 × 100 or 400 × 400 Dalitz plots. Use 18 positive time and 18 negative time bins, starting with 0.25 lifetime width.

Additional Comments

- We assume CP symmetry
- Results are generally insensitive to Dalitz plot binning
- Results are generally insensitive to time binning
- $XX^*$ and $YY^*$ can usually be extracted from time-integrated data with minimal assumptions
- $\Re(XY^*)$ and $\Im(XY^*)$ can probably be extracted from time-integrated data, but it will be more difficult.
## Sensitivities

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<tr>
<th>channel</th>
<th># of events</th>
<th>$\delta x$</th>
<th>$\delta y$</th>
<th>Comments</th>
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<td>720K</td>
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<td>0.09%</td>
<td>BaBar $K^0_S\pi^-\pi^+$ amplitudes</td>
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<td>0.05%</td>
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<td>–</td>
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</tr>
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<td>0.06%</td>
<td>$\cos \delta_{K\pi\pi^0} = 0.95$, $R_D = 0.16%$</td>
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<td>BaBar $K\pi\pi^0$ amplitudes</td>
</tr>
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<td>–</td>
<td>0.16%</td>
<td>$\cos \delta_{K\pi} = 0.95$</td>
</tr>
<tr>
<td>$h^-h^+, K^-e^+\nu_e$</td>
<td>345K</td>
<td>–</td>
<td>0.12%</td>
<td></td>
</tr>
<tr>
<td>$\pi^-\pi^+\pi^0, K^-e^+\nu_e$</td>
<td>120K</td>
<td>0.28%</td>
<td>0.22%</td>
<td>BaBar $\pi^-\pi^+\pi^0$ amplitudes</td>
</tr>
<tr>
<td>$\pi^-\pi^+\pi^0, K^-\pi^+$</td>
<td>120K</td>
<td>0.56%</td>
<td>0.15%</td>
<td></td>
</tr>
<tr>
<td>$\pi^-\pi^+\pi^0, h^-h^+$</td>
<td>20K</td>
<td>–</td>
<td>0.5%</td>
<td></td>
</tr>
</tbody>
</table>
CPV in Time Asymmetric Rates

The time asymmetric part of the correlated rate depends on

$$X = \frac{q}{p} \bar{A}_\alpha \bar{A}_\beta - \frac{p}{q} A_\alpha A_\beta ; \quad Y = A_\alpha \bar{A}_\beta - \bar{A}_\alpha A_\beta$$  \hspace{1cm} (64)

through

$$4|\mathcal{M}|^2 = e^{-\Gamma(t_1+t_2)} \times \left\{ \begin{array}{l} XX^* \left[ \frac{(x^2 + y^2)}{4} (\Gamma \Delta t)^2 \right] \\ - \Re(XY^*) y \Gamma \Delta t + \Im(XY^*) x \Gamma \Delta t \\ + YY^* \left[ 1 + \frac{(y^2 - x^2)}{4} (\Gamma \Delta t)^2 \right] \end{array} \right\}$$  \hspace{1cm} (65)

In the limit of no direct CPV, $A_\alpha \rightarrow \bar{A}_\alpha$; $A_\beta \rightarrow \bar{A}_\beta$ and $q/p \rightarrow p/q$ when we interchange $D^0$ and $\bar{D}^0$. This leads to CPV in the time-odd rate proportional to the real and imaginary parts of $XY^*$. As an example, recall that for ($K^-\pi^+, K^-\pi^+\pi^0$),

$$XY^* \approx -\frac{p}{q} \left\{ \begin{array}{l} [k_2 \cos \delta_2 \Re(\zeta \bar{\zeta}^*) + k_2 \sin \delta_2 \Im(\zeta \bar{\zeta}^*) - k_1 \cos \delta_1 \zeta \bar{\zeta}^*] \\ + i [k_2 \cos \delta_2 \Im(\zeta \bar{\zeta}^*) - k_2 \sin \delta_2 \Re(\zeta \bar{\zeta}^*) + k_1 \sin \delta_1 \zeta \bar{\zeta}^*] \end{array} \right\} |A_\alpha|^2 |A_r|^2.$$  \hspace{1cm} (66)
Time-Odd CPV Sensitivities

Some observations:

- \( \delta x \) and \( \delta y \) sensitivities somewhat worse when allowing for CPV;
- \( \delta x \) and \( \delta y \) sensitivities independent of central values;
- \( \delta x \) and \( \delta y \) sensitivities scale like \( 1/\sqrt{n} \);
- \( \delta |q/p| \) and \( \delta \phi \) sensitivities depend on central values of \( x \) and \( y \);
- \( \delta |q/p| \) and \( \delta \phi \) sensitivities scale like \( 1/\sqrt{n} \).
Quick Summary

• Mixing has been observed at the 10 σ level with $x$ and $y \sim 1\%$. No CPV has been observed, with the error on $|q/p| \sim 10\%$.

• Observations are possibly consistent with the Standard Model or New Physics amplitudes, or a combination of both.

• With 500 fb$^{-1}$ of data at charm threshold, as anticipated in less than a year’s running at SuperB, correlated decays have the potential to reduce the errors on mixing and CPV parameters by an order of magnitude. Observation of CPV in mixing at this level would be a clear signal of New Physics. If New Physics has already been observed by ATLAS and CMS, it will constrain the NP flavor couplings.